

d'Alembert-type scheme with a chain regularization for N -body problem

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ABSTRACT

We design an accurate orbital integration scheme for the general N -body problem preserving all the conserved quantities but the angular momentum. This scheme is based on the chain concept (Mikkola & Aarseth 1993) and is regarded as an extension of a d’Alembert-type scheme (Betsch 2005) for constrained Hamiltonian systems. It also coincides with the discrete-time general three-body problem (Minesaki 2013a) for particle number $N = 3$. Although the proposed scheme is only second-order accurate, it can accurately reproduce some periodic orbits, which generic geometric numerical integrators cannot do.

Subject headings: celestial mechanics - methods: numerical

1. Introduction

To find periodic orbits in the N -body problem ($N \geq 3$), we use methods (e.g., Baltagiannis & Papadakis (2011b); Broucke (1969)) consisting of two procedures: (1) We introduce a rotating-pulsating frame where a few primaries are fixed. (2) We use the Runge–Kutta–Fehlberg method and set the allowable energy variation and errors of the positions, or the Steffensen method with recurrent power series so that we compute the periodic orbits in one period. However, these methods are not suitable for reproducing the orbits in this problem for a long time interval because of the following two drawbacks: (a) A rotating-pulsating frame, in which the methods described in (2) are applied, has to be altered in accordance with periodic orbits. (b) The methods in (2) cannot accurately compute periodic orbits for a long time interval because they do not preserve any conserved quantities.

On the other hand, for any initial condition, including the conditions of some periodic orbits, numerical integration methods are applied to the N -body problem in the barycentric inertial frame. If we use a non-geometric integration method, this method cannot reproduce periodic orbits for a long time interval because of drawback (b). In addition, even if we used each of the geometric integration methods (e.g., the symplectic and energy-momentum methods), they cannot necessarily reproduce periodic orbits. Both the symplectic and energy-momentum methods cannot illustrate elliptic orbits in the two-body problem (Minesaki 2002, 2004) and elliptic Lagrange orbits in the three-body problem (Minesaki 2013a). To overcome drawbacks (a) and (b), the author already proposed the discrete-time general three-body problem (d-G3BP) (Minesaki 2013a) and the discrete-time restricted three-body problem (d-R3BP) (Minesaki 2013c) for the general three-body problem (G3BP) and restricted three-body problem (R3BP) in the barycentric inertial frame, respectively. These schemes (Minesaki 2013a,c) are given by an extension of

a d’Alembert-type scheme (Betsch 2005). The d-G3BP retains all the conserved quantities but the angular momentum, and the d-R3BP preserves all the conserved quantities but the Jacobi integration. In this paper, we design an accurate orbital integration scheme like the d-G3BP and d-R3BP for the general N -body problem (GNBP). The new scheme is based on a d’Alembert-type scheme (Betsch 2005) and a chain regularization (Mikkola & Aarseth 1993). It keeps all the conserved quantities except the angular momentum and can accurately compute some periodic orbits.

This paper is organized as follows. In Section 2, after labeling the masses according to the chain concept (Mikkola & Aarseth 1993) and using the Levi-Civita transformation (Levi-Civita 1920), we express the general N -body problem as a constrained Hamiltonian system without Lagrangian multipliers. Further, we rewrite this problem using only the vectors related to the chained ones. In Section 3, we apply the same discrete-time formulation adopted for the G3BP in (Minesaki 2013a) to the resulting problem, so we have a discrete-time problem. We prove that the discrete-time problem preserves all the conserved quantities of the GNBP except the angular momentum. In Section 4, we check that the discrete-time problem ensures such preservation of the general N -body problem numerically. Moreover, we show that it correctly calculates some periodic orbits.

2. Regularization of General N -body Problem

For an arbitrary number of masses N , we give the transformation formulae, equations of motion for the GNBP, and selection of a chain of interparticle vectors such that the close encounters requiring regularization are included in the chain. This formulation includes the same transformation formulae and selection of a chain as in (Mikkola & Aarseth 1993). It has the advantage that its computational cost is far lower than that of Heggie’s global formulation (Heggie 1974) for a large number of masses N .

In Section 2.1, we briefly review the GNB in the barycentric frame and how to form a chain of interparticle vectors and label masses using the chain algorithm in (Mikkola & Aarseth 1993). In Section 2.2, using the Levi-Civita transformation (Levi-Civita 1920), we rewrite the GNB, which is similar to the problem given by Heggie’s global regularization (Heggie 1974). For a large number of masses N , the rewritten problem involves many redundant variables. In Section 2.3, using some constraints, we express the problem in terms of only the chained position and momentum vectors to reduce the number of redundant variables.

2.1. Labeling Particles Using Chain Concept

The small distance between two bodies experiencing a close encounter is represented as a difference between large numbers in straightforward formulations of the N -body problem. Thus, round-off easily becomes a significant source of error. To avoid this, we use the chain concept of (Mikkola & Aarseth 1993) introduced for regularization algorithms.

In this chain method, a chain of interparticle vectors is constructed so that all the particles are included in this chain. Note that small distances are part of the chain. We begin by searching for the shortest distance, which is taken as the first piece of the chain. Next, we find the particle closest to one or the other end of the presently known part of the chain. Then, we add this particle to the end of the chain that is closer. This process is repeated until all the particles are involved. After every integration step, we check whether any non-chained vector is shorter than the smallest of the chained vectors that are in contact with one or the other end of the vector under consideration, namely, if any triangle formed by two consecutive chain vectors has the shortest side non-chained. If this is the case, a new chain is formed. Hereafter, suppose the masses are relabeled $1, 2, \dots, N$ along the chain.

We assume that $\mathbf{q}'_i \equiv (q'_{i[1]}, q'_{i[2]})$ is the position vector of a point with mass m_i in the barycentric frame. We also define $\mathbf{p}'_i \equiv (p'_{i[1]}, p'_{i[2]})$ as a momentum conjugate to \mathbf{q}'_i . We set the gravitational constant equal to one for simplicity. In addition, N position vectors $\mathbf{q}'_1, \dots, \mathbf{q}'_N$ satisfy the following constraints:

$$\sum_{i=1}^N m_i \mathbf{q}'_i = \mathbf{0}.$$

The equations of motion in the barycentric frame are given by the Hamiltonian:

$$H = \frac{1}{2} \sum_{i=1}^N \frac{|\mathbf{p}'_i|^2}{m_i} - \sum_{i=1}^{N-1} \sum_{j=i+1}^N \frac{m_i m_j}{|\mathbf{q}'_i - \mathbf{q}'_j|}. \quad (1)$$

The dynamical system corresponding to this Hamiltonian is

$$\frac{d}{dt} \mathbf{q}'_i = \frac{1}{m_i} \mathbf{p}'_i, \quad \frac{d}{dt} \mathbf{p}'_i = m_i \left(\sum_{k=1}^{i-1} \frac{m_k (\mathbf{q}'_k - \mathbf{q}'_i)}{|\mathbf{q}'_k - \mathbf{q}'_i|^3} - \sum_{k=i+1}^N \frac{m_k (\mathbf{q}'_i - \mathbf{q}'_k)}{|\mathbf{q}'_i - \mathbf{q}'_k|^3} \right), \quad 1 \leq i \leq N. \quad (2)$$

However, for two-body close encounters, we need to simultaneously use two position vectors in the barycentric frame. Therefore, the barycentric frame is not useful for computing close encounters between two masses.

2.2. General N -body Problem with Redundant Variables

The GNBP in the relative frame is much more symmetric than that in the barycentric frame. It also has a significant advantage in investigating such properties as periodic orbits and close encounters (e.g., (Broucke 1975; Broucke & Boggs 1975; Mikkola & Tanikawa 1999)). It can be integrated numerically without catastrophic errors after the Levi-Civita or Kustaanheimo–Stiefel transformation (Kustaanheimo & Stiefel 1965; Levi-Civita 1920; Stiefel & Scheifele 1971).

In Section 2.2.1, we rewrite the GNBP in the relative frame. The resulting problem involves very many gravitational force terms for a large number of masses N . Thus, we

deform this system to reduce the number of force terms. In Section 2.2.2, we rewrite the system using the Levi-Civita variables.

2.2.1. General N -body Problem in Relative Frame

We introduce a relative frame to consider two-body close approaches easily. We use the relative position vectors $\mathbf{q}_{ij} \equiv (q_{ij[1]}, q_{ij[2]})$ defined by

$$\mathbf{q}_{ij} = \mathbf{q}'_i - \mathbf{q}'_j, \quad 1 \leq i < j \leq N, \quad (3)$$

and the momentum $\mathbf{p}_{ij} \equiv (p_{ij[1]}, p_{ij[2]})$ conjugate to \mathbf{q}_{ij} as

$$\mathbf{p}_{ij} = \frac{m_j \mathbf{p}'_i - m_i \mathbf{p}'_j}{m}, \quad 1 \leq i < j \leq N, \quad (4)$$

where m is the total mass, $\sum_{i=1}^N m_i$, of the GNB. These position and momentum vectors also satisfy the following constraints:

$$\begin{cases} \phi_{1jk}(\mathbf{q}) \equiv (\phi_{1jk[1]}(\mathbf{q}), \phi_{1jk[2]}(\mathbf{q})) \equiv \mathbf{q}_{1j} + \mathbf{q}_{jk} - \mathbf{q}_{1k} = \mathbf{0}, & 2 \leq j < k \leq N, \\ \psi_{ijk}(\mathbf{p}) \equiv (\psi_{ijk[1]}(\mathbf{p}), \psi_{ijk[2]}(\mathbf{p})) \equiv \frac{\mathbf{p}_{ij}}{m_i m_j} + \frac{\mathbf{p}_{jk}}{m_j m_k} - \frac{\mathbf{p}_{ik}}{m_i m_k} = \mathbf{0}, & 1 \leq i < j < k \leq N. \end{cases} \quad (5a) \quad (5b)$$

Here, $\mathbf{q} = (q_{1,2[1]}, q_{1,2[2]}, \begin{smallmatrix} \vdots \\ \vdots \end{smallmatrix} \dots \begin{smallmatrix} \vdots \\ \vdots \end{smallmatrix} q_{1,N[1]}, q_{1,N[2]}, \left| \begin{smallmatrix} \vdots \\ \vdots \end{smallmatrix} q_{2,3[1]}, q_{2,3[2]}, \begin{smallmatrix} \vdots \\ \vdots \end{smallmatrix} \dots \begin{smallmatrix} \vdots \\ \vdots \end{smallmatrix} q_{2,N[1]}, q_{2,N[2]}, \left| \dots \left| \begin{smallmatrix} \vdots \\ \vdots \end{smallmatrix} q_{N-1,N[1]}, q_{N-1,N[2]} \right. \right) \in \mathbb{R}^{N(N-1)}$ and

$\mathbf{p} = (p_{1,2[1]}, p_{1,2[2]}, \begin{smallmatrix} \vdots \\ \vdots \end{smallmatrix} \dots \begin{smallmatrix} \vdots \\ \vdots \end{smallmatrix} p_{1,N[1]}, p_{1,N[2]}, \left| \begin{smallmatrix} \vdots \\ \vdots \end{smallmatrix} p_{2,3[1]}, p_{2,3[2]}, \begin{smallmatrix} \vdots \\ \vdots \end{smallmatrix} \dots \begin{smallmatrix} \vdots \\ \vdots \end{smallmatrix} p_{2,N[1]}, p_{2,N[2]}, \left| \dots \left| \begin{smallmatrix} \vdots \\ \vdots \end{smallmatrix} p_{N-1,N[1]}, p_{N-1,N[2]} \right. \right) \in \mathbb{R}^{N(N-1)}.$

To obtain the inverse transformations of equations (3) and (4), we have to solve system (3) for the position vectors \mathbf{q}'_i and system (4) for the momentum vectors \mathbf{p}'_i . Unfortunately, no vector is uniquely determined. However, if we choose

$$\mathbf{q}'_i = \frac{1}{m} \left(\sum_{j=i+1}^N m_j \mathbf{q}_{ij} - \sum_{j=1}^{i-1} m_j \mathbf{q}_{ji} \right), \quad \mathbf{p}'_i = \sum_{j=i+1}^N \mathbf{p}_{ij} - \sum_{j=1}^{i-1} \mathbf{p}_{ji}, \quad 1 \leq i \leq N, \quad (6)$$

then these relations follow equations (3) and (4). Accordingly, we adopt equation (6) as a transformation from the relative frame to the barycentric frame.

Substitution of transformations (3) and (4) and their time differentiations into equation (2) yields the following system:

$$\begin{cases} \frac{d}{dt}\mathbf{q}_{ij} = \frac{m}{m_i m_j} \mathbf{p}_{ij}, & 1 \leq i < j \leq N, \end{cases} \quad (7a)$$

$$\begin{cases} \frac{d}{dt}\mathbf{p}_{1j} = -m_1 m_j \frac{\mathbf{q}_{1j}}{|\mathbf{q}_{1j}|^3} + \sum_{k=2}^{j-1} \mathbf{f}_{kj}(\mathbf{q}) - \sum_{k=j+1}^N \mathbf{f}_{jk}(\mathbf{q}), & 2 \leq j \leq N, \end{cases} \quad (7b)$$

$$\begin{cases} \frac{d}{dt}\mathbf{p}_{ij} = -m_i m_j \frac{\mathbf{q}_{ij}}{|\mathbf{q}_{ij}|^3} - \mathbf{f}_{ij}(\mathbf{q}), & 2 \leq i < j \leq N, \end{cases} \quad (7c)$$

where $\mathbf{f}_{ij}(\mathbf{q})$ is a function of a vector \mathbf{q} defined by

$$\begin{aligned} \mathbf{f}_{ij}(\mathbf{q}) \equiv & -\sum_{k=1}^{i-1} \frac{m_i m_j m_k}{m} \left(\frac{\mathbf{q}_{ki}}{|\mathbf{q}_{ki}|^3} + \frac{\mathbf{q}_{ij}}{|\mathbf{q}_{ij}|^3} - \frac{\mathbf{q}_{kj}}{|\mathbf{q}_{kj}|^3} \right) + \sum_{k=i+1}^{j-1} \frac{m_i m_j m_k}{m} \left(\frac{\mathbf{q}_{ik}}{|\mathbf{q}_{ik}|^3} + \frac{\mathbf{q}_{kj}}{|\mathbf{q}_{kj}|^3} - \frac{\mathbf{q}_{ij}}{|\mathbf{q}_{ij}|^3} \right) \\ & - \sum_{k=j+1}^N \frac{m_i m_j m_k}{m} \left(\frac{\mathbf{q}_{ij}}{|\mathbf{q}_{ij}|^3} + \frac{\mathbf{q}_{jk}}{|\mathbf{q}_{jk}|^3} - \frac{\mathbf{q}_{ik}}{|\mathbf{q}_{ik}|^3} \right), \quad 2 \leq i < j \leq N. \end{aligned} \quad (8)$$

We can regard the system composed of equations (5a) and (7) as the GNPB in the relative frame.

Through equation (6), the Hamiltonian (1) leads to

$$H = \sum_{i=1}^{N-1} \sum_{j=i+1}^N \left(\frac{m}{2m_i m_j} |\mathbf{p}_{ij}|^2 - \frac{m_i m_j}{|\mathbf{q}_{ij}|} \right) + \frac{1}{2} \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} \sum_{k=j+1}^N m_i m_j m_k |\boldsymbol{\psi}_{ijk}(\mathbf{p})|^2.$$

However, it does not yield system (7), so it cannot be regarded as a Hamiltonian. System (7) is governed by the following Hamiltonian:

$$H_{\text{rel}} \equiv \sum_{i=1}^{N-1} \sum_{j=i+1}^N \left(\frac{m}{2m_i m_j} |\mathbf{p}_{ij}|^2 - \frac{m_i m_j}{|\mathbf{q}_{ij}|} \right) + \sum_{j=2}^{N-1} \sum_{k=j+1}^N \boldsymbol{\phi}_{1jk}(\mathbf{q}) \cdot \boldsymbol{\lambda}_{jk}, \quad (9)$$

where $\boldsymbol{\lambda}_{jk} = (\lambda_{jk[1]}, \lambda_{jk[2]})$ are Lagrange multipliers¹. The values of H and H_{rel} coincide

¹ Using the second time derivative of the constraints (5a), namely,

with

$$h_{\text{rel}} \equiv \sum_{i=1}^{N-1} \sum_{j=i+1}^N \left(\frac{m}{2m_i m_j} |\mathbf{p}_{ij}|^2 - \frac{m_i m_j}{|\mathbf{q}_{ij}|} \right). \quad (10)$$

2.2.2. General N -body Problem with Levi-Civita Variables

In the numerical integration of the GNB, multibody close encounters result in serious numerical difficulties due to the errors associated with singularities in the GNB. The Levi-Civita regularization is a standard technique for removing the singularities. It combines a time regularization with the Levi-Civita transformation (Levi-Civita 1920). In this subsection, using the Levi-Civita variables (Levi-Civita 1920), we rewrite the GNB (7).

We apply the Levi-Civita transformation (Levi-Civita 1920) to the vectors \mathbf{q}_{ij} , obtaining the vectors $\mathbf{Q}_{ij} \equiv (Q_{ij[1]}, Q_{ij[2]})$. The relations between \mathbf{q}_{ij} and \mathbf{Q}_{ij} are given by

$$\mathbf{q}_{ij} = \mathbf{Q}_{ij} \mathbf{L}(\mathbf{Q}_{ij})^\top, \quad 1 \leq i < j \leq N, \quad (11)$$

where the Levi-Civita matrix (Levi-Civita 1920) is defined as

$$\mathbf{L}(\mathbf{Q}_{ij}) \equiv \begin{bmatrix} Q_{ij[1]} & -Q_{ij[2]} \\ Q_{ij[2]} & Q_{ij[1]} \end{bmatrix}.$$

New momentum vectors $\mathbf{P}_{ij} \equiv (P_{ij[1]}, P_{ij[2]})$ are also related to the old ones \mathbf{p}_{ij} by the equations

$$\mathbf{p}_{ij} = \frac{1}{2|\mathbf{Q}_{ij}|^2} \mathbf{P}_{ij} \mathbf{L}(\mathbf{Q}_{ij})^\top, \quad 1 \leq i < j \leq N. \quad (12)$$

$d(\mathbf{p}_{1j}/m_1 m_j + \mathbf{p}_{jk}/m_j m_k - \mathbf{p}_{1k}/m_1 m_k)/dt = \mathbf{0}$ ($2 \leq j < k \leq N$), we can give the Lagrange multipliers as $\boldsymbol{\lambda}_{jk} = \mathbf{f}_{jk}(\mathbf{q})$. Therefore, system (7) is governed by (9).

Theoretically, we can use the solution of relations (11) and (12) to obtain \mathbf{Q}_{ij} and \mathbf{P}_{ij} from \mathbf{q}_{ij} and \mathbf{p}_{ij} . In numerical computation, to avoid cancellation of significant digits, we compute \mathbf{Q}_{ij} and \mathbf{P}_{ij} for $1 \leq i < j \leq N$ as follows:

$$\mathbf{Q}_{ij} \equiv \begin{cases} \left(\frac{q_{ij[2]}}{\sqrt{-2q_{ij[1]} + 2|\mathbf{q}_{ij}|}}, \frac{1}{2}\sqrt{-2q_{ij[1]} + 2|\mathbf{q}_{ij}|} \right) & \text{if } q_{ij[1]} < 0, \\ \left(\frac{1}{2}\sqrt{2q_{ij[1]} + 2|\mathbf{q}_{ij}|}, \frac{q_{ij[2]}}{\sqrt{2q_{ij[1]} + 2|\mathbf{q}_{ij}|}} \right) & \text{if } q_{ij[1]} \geq 0, \end{cases} \quad (13)$$

$$\mathbf{P}_{ij} \equiv \begin{cases} \left(\frac{2(p_{ij[1]}q_{ij[2]} - p_{ij[2]}q_{ij[1]} + p_{ij[2]}|\mathbf{q}_{ij}|)}{\sqrt{-2q_{ij[1]} + 2|\mathbf{q}_{ij}|}}, \frac{2(p_{ij[1]}q_{ij[1]} + p_{ij[2]}q_{ij[2]} - p_{ij[1]}|\mathbf{q}_{ij}|)}{\sqrt{-2q_{ij[1]} + 2|\mathbf{q}_{ij}|}} \right) & \text{if } q_{ij[1]} < 0, \\ \left(\frac{2(p_{ij[1]}q_{ij[1]} + p_{ij[2]}q_{ij[2]} + p_{ij[1]}|\mathbf{q}_{ij}|)}{\sqrt{2q_{ij[1]} + 2|\mathbf{q}_{ij}|}}, -\frac{2(p_{ij[1]}q_{ij[2]} - p_{ij[2]}q_{ij[1]} - p_{ij[2]}|\mathbf{q}_{ij}|)}{\sqrt{2q_{ij[1]} + 2|\mathbf{q}_{ij}|}} \right) & \text{if } q_{ij[1]} \geq 0. \end{cases} \quad (14)$$

Substitution of transformations (11) and (12) and their time differentiations into equation (7) yields

$$\begin{cases} \mathbf{F}_{ij} = \mathbf{0}, & 1 \leq i < j \leq N, \end{cases} \quad (15a)$$

$$\begin{cases} \mathbf{G}_{1j} - \sum_{k=2}^{j-1} \boldsymbol{\lambda}_{kj} + \sum_{k=j+1}^N \boldsymbol{\lambda}_{jk} = \mathbf{0}, & 2 \leq j \leq N, \end{cases} \quad (15b)$$

$$\begin{cases} \mathbf{G}_{ij} + \boldsymbol{\lambda}_{ij} = \mathbf{0}, & 1 < i < j \leq N, \end{cases} \quad (15c)$$

where

$$\begin{aligned} \mathbf{F}_{ij} &\equiv \frac{d}{dt}\mathbf{Q}_{ij} - \frac{m}{4m_i m_j |\mathbf{Q}_{ij}|^2}, \quad 1 \leq i < j \leq N, \\ \mathbf{G}_{ij} &\equiv \frac{1}{2|\mathbf{Q}_{ij}|^2} \left(\frac{d}{dt}\mathbf{P}_{ij} - \left(\frac{m}{4m_i m_j} |\mathbf{P}_{ij}|^2 - 2m_i m_j \right) \frac{\mathbf{Q}_{ij}}{|\mathbf{Q}_{ij}|^4} \right) \mathbf{L}^\top(\mathbf{Q}_{ij}), \quad 1 \leq i < j \leq N, \\ \boldsymbol{\lambda}_{ij} &\equiv - \sum_{k=1}^{i-1} \frac{m_i m_j m_k}{m} \left(\frac{\mathbf{Q}_{ki}}{|\mathbf{Q}_{ki}|^6} \mathbf{L}^\top(\mathbf{Q}_{ki}) + \frac{\mathbf{Q}_{ij}}{|\mathbf{Q}_{ij}|^6} \mathbf{L}^\top(\mathbf{Q}_{ij}) - \frac{\mathbf{Q}_{kj}}{|\mathbf{Q}_{kj}|^6} \mathbf{L}^\top(\mathbf{Q}_{kj}) \right) \\ &\quad + \sum_{k=i+1}^{j-1} \frac{m_i m_j m_k}{m} \left(\frac{\mathbf{Q}_{ik}}{|\mathbf{Q}_{ik}|^6} \mathbf{L}^\top(\mathbf{Q}_{ik}) + \frac{\mathbf{Q}_{kj}}{|\mathbf{Q}_{kj}|^6} \mathbf{L}^\top(\mathbf{Q}_{kj}) - \frac{\mathbf{Q}_{ij}}{|\mathbf{Q}_{ij}|^6} \mathbf{L}^\top(\mathbf{Q}_{ij}) \right) \\ &\quad - \sum_{k=j+1}^N \frac{m_i m_j m_k}{m} \left(\frac{\mathbf{Q}_{ij}}{|\mathbf{Q}_{ij}|^6} \mathbf{L}^\top(\mathbf{Q}_{ij}) + \frac{\mathbf{Q}_{jk}}{|\mathbf{Q}_{jk}|^6} \mathbf{L}^\top(\mathbf{Q}_{jk}) - \frac{\mathbf{Q}_{ik}}{|\mathbf{Q}_{ik}|^6} \mathbf{L}^\top(\mathbf{Q}_{ik}) \right), \quad 2 \leq i < j \leq N. \end{aligned} \quad (16)$$

In addition, through equation (11), the constraints in equation (5a) lead to

$$\Phi_{1jk}(\mathbf{Q}) \equiv \begin{bmatrix} \Phi_{1jk[1]}(\mathbf{Q}) \\ \Phi_{1jk[2]}(\mathbf{Q}) \end{bmatrix}^\top \equiv \mathbf{Q}_{1j} \mathbf{L}(\mathbf{Q}_{1j})^\top + \mathbf{Q}_{jk} \mathbf{L}(\mathbf{Q}_{jk})^\top - \mathbf{Q}_{1k} \mathbf{L}(\mathbf{Q}_{1k})^\top = \mathbf{0}, \quad 2 \leq j < k \leq N, \quad (17)$$

where $\mathbf{Q} = (\mathbf{Q}_{1,2}, \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \dots \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \mathbf{Q}_{1,N}, \left| \mathbf{Q}_{2,3}, \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \dots \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \mathbf{Q}_{2,N}, \left| \dots \right| \mathbf{Q}_{N-1,N}) \in \mathbb{R}^{N(N-1)}$ is a position vector. The system composed of equations (15) and (17) represents the motion of the GNBP, which is described by the Levi-Civita variables and the Lagrange multipliers λ_{jk} . This system is governed by the following Hamiltonian:

$$H_{\text{LC}} \equiv \sum_{i=1}^{N-1} \sum_{j=i+1}^N \left(\frac{m}{8m_i m_j} \frac{|\mathbf{P}_{ij}|^2}{|\mathbf{Q}_{ij}|^2} - \frac{m_i m_j}{|\mathbf{Q}_{ij}|^2} \right) + \sum_{j=2}^{N-1} \sum_{k=j+1}^N \Phi_{1jk}(\mathbf{Q}) \cdot \lambda_{jk}, \quad (18)$$

which is obtained by substituting equations (11) and (12) into H_{rel} defined by equation (9).

Using equation (15c), we rewrite equation (15b) as the following identities:

$$\sum_{i=1}^{j-1} \mathbf{G}_{ij} - \sum_{i=j+1}^N \mathbf{G}_{ji} = \mathbf{0}, \quad 2 \leq j \leq N. \quad (19)$$

The new system composed of equations (15a), (17), and (19) describes the same motion as the system composed of equations (15) and (17). The number of dependent variables, $P_{ij[1]}$, $P_{ij[2]}$, $Q_{ij[1]}$, and $Q_{ij[2]}$ in the new system is $2N(N-1)$ rather than $4N$, which is the number of dependent variables, $p'_{i[1]}$, $p'_{i[2]}$, $q'_{i[1]}$, and $q'_{i[2]}$, in system (2). Actually, as the total number of masses N increases, the number of equations in the new system increases more rapidly. Therefore, the computational cost of the new system is very high for large N . We call this system the *redundant general three-body problem (RGNBP)*.

H_{LC} is conserved by the RGNBP as well as the system composed of equations (15) and (17). Because of equation (17), the value of H_{LC} defined by equation (18) is equivalent to

$$h_{\text{LC}} = \sum_{i=1}^{N-1} \sum_{j=i+1}^N \left(\frac{m}{8m_i m_j} \frac{|\mathbf{P}_{ij}|^2}{|\mathbf{Q}_{ij}|^2} - \frac{m_i m_j}{|\mathbf{Q}_{ij}|^2} \right). \quad (20)$$

Further, h_{LC} equals the value of H defined by equation (1) because h_{rel} defined by (10) is transformed to h_{LC} through equations (11) and (12), and h_{rel} is the value of H as described in Section 2.2.1.

2.3. Chain Regularization of General N -body Problem

In this section, we rewrite the RGNBP using only $\mathbf{P}_{k,k+1}$ and $\mathbf{Q}_{k,k+1}$ ($1 \leq k \leq N-1$) so that we reduce the redundancy of the problem, which incurs a high computational cost. The vectors $\mathbf{Q}_{k,k+1}$ and $\mathbf{P}_{k,k+1}$ are related to the chained position vectors $\mathbf{q}_{k,k+1}$ and their momenta $\mathbf{p}_{k,k+1}$ conjugate to $\mathbf{q}_{k,k+1}$, respectively.

First, we express the vectors \mathbf{Q}_{ij} related to the non-chained position vectors \mathbf{q}_{ij} ($1 \leq i < i+2 \leq j \leq N$) as functions of $\mathbf{Q}_{k,k+1}$ related to the chained position vectors $\mathbf{q}_{k,k+1}$ ($1 \leq k \leq N-1$). The constraints in equation (17) restrict the possible positions of the RGNBP to the $(N-1) - (N-1)(N-2) = 2(N-1)$ -dimensional configuration manifold

$$\mathcal{Q} = \left\{ \mathbf{Q} \in \mathbb{R}^{N(N-1)} \mid \Phi(\mathbf{Q}) = \mathbf{0}_{1 \times (N-1)(N-2)} \right\}, \quad (21)$$

where $\Phi(\mathbf{Q}) = (\Phi_{1,2,3}(\mathbf{Q}), \dots, \Phi_{1,2,N}(\mathbf{Q}), \left| \Phi_{1,3,4}(\mathbf{Q}), \dots, \Phi_{1,3,N}(\mathbf{Q}), \dots \right| \Phi_{1,N-1,N}(\mathbf{Q})) \in \mathbb{R}^{(N-1)(N-2)}$ is a constraint function vector. Accordingly, the set of $2(N-1)$ variables $\left\{ Q_{1,2[1]}, Q_{1,2[2]}, \dots, Q_{2,3[1]}, Q_{2,3[2]}, \dots, Q_{N-1,N[1]}, Q_{N-1,N[2]} \right\}$, which corresponds to the $(N-1)$ vectors $\mathbf{Q}_{k,k+1}$ ($1 \leq k \leq N-1$), acts as a basis for the manifold \mathcal{Q} . Then, every vector $\mathbf{Q}_{ij} \in \mathbb{R}^{(N-1)(N-2)}$ ($1 \leq i < i+2 \leq j \leq N$) related to a non-chained position vector \mathbf{q}_{ij} fulfills the following relation:

$$\mathbf{Q}_{ij} \mathbf{L}(\mathbf{Q}_{ij})^\top = \sum_{k=i}^{j-1} \mathbf{Q}_{k,k+1} \mathbf{L}(\mathbf{Q}_{k,k+1})^\top, \quad 1 \leq i < i+2 \leq j \leq N. \quad (22)$$

The solutions of equation (22) for \mathbf{Q}_{ij} are

$$\mathbf{Q}_{ij} = \tilde{\mathbf{Q}}_{ij} \equiv \left(\sqrt{\frac{a_{ij} + b_{ij}}{2}}, \frac{\sqrt{2} \sum_{k=i}^{j-1} Q_{k,k+1[1]} Q_{k,k+1[2]}}{\sqrt{a_{ij} + b_{ij}}} \right), \quad 1 \leq i < i+2 \leq j \leq N, \quad (23)$$

where

$$a_{ij} \equiv \sqrt{\left(\sum_{k=i}^{j-1} (Q_{k,k+1[1]}^2 - Q_{k,k+1[2]}^2) \right)^2 + 4 \left(\sum_{k=i}^{j-1} Q_{k,k+1[1]} Q_{k,k+1[2]} \right)^2},$$

$$b_{ij} \equiv \sum_{k=i}^{j-1} (Q_{k,k+1[1]}^2 - Q_{k,k+1[2]}^2), \quad 1 \leq i < i+2 \leq j \leq N.$$

Next, we write the vectors \mathbf{P}_{ij} ($1 \leq i < i+2 \leq j \leq N$) related to the momentum vectors \mathbf{p}_{ij} as functions of $\mathbf{Q}_{k,k+1}$ and $\mathbf{P}_{k,k+1}$ ($1 \leq k \leq N-1$) related to the momentum vectors $\mathbf{p}_{k,k+1}$. The momenta of the RGNBP, \mathbf{P}_{jk} , satisfy the following constraints:

$$\Psi_{1jk}(\mathbf{P}, \mathbf{Q}) \equiv \frac{1}{m_1 m_j} \frac{\mathbf{P}_{1j} \mathbf{L}(\mathbf{Q}_{1j})^\top}{|\mathbf{Q}_{1j}|^2} + \frac{1}{m_j m_k} \frac{\mathbf{P}_{jk} \mathbf{L}(\mathbf{Q}_{jk})^\top}{|\mathbf{Q}_{jk}|^2} - \frac{1}{m_1 m_k} \frac{\mathbf{P}_{1k} \mathbf{L}(\mathbf{Q}_{1k})^\top}{|\mathbf{Q}_{1k}|^2} = \mathbf{0}, \quad 2 \leq j < k \leq N, \quad (24)$$

which is obtained by substituting equation (15a) into the time derivative of $\Phi_{1jk}(\mathbf{Q})$. Here,

$\mathbf{P} = (\mathbf{P}_{1,2}, \begin{vmatrix} \vdots \\ \vdots \end{vmatrix} \cdots \begin{vmatrix} \vdots \\ \vdots \end{vmatrix} \mathbf{P}_{1,N}, \begin{vmatrix} \vdots \\ \vdots \end{vmatrix} \mathbf{P}_{2,3}, \begin{vmatrix} \vdots \\ \vdots \end{vmatrix} \cdots \begin{vmatrix} \vdots \\ \vdots \end{vmatrix} \mathbf{P}_{2,N}, \begin{vmatrix} \vdots \\ \vdots \end{vmatrix} \cdots \begin{vmatrix} \vdots \\ \vdots \end{vmatrix} \mathbf{P}_{N-1,N}) \in \mathbb{R}^{N(N-1)}$ is a momentum vector.

The constraints in equation (24) restrict the possible momenta of the RGNBP to the $N(N-1) - (N-1)(N-2) = 2(N-1)$ -dimensional manifold

$$\mathcal{P} = \{ \mathbf{P} \in \mathbb{R}^{N(N-1)} \mid \Psi(\mathbf{P}, \mathbf{Q}) = \mathbf{0}_{1 \times (N-1)(N-2)} \},$$

where $\Psi(\mathbf{P}, \mathbf{Q}) = (\Psi_{1,2,3}(\mathbf{P}, \mathbf{Q}), \begin{vmatrix} \vdots \\ \vdots \end{vmatrix} \cdots \begin{vmatrix} \vdots \\ \vdots \end{vmatrix} \Psi_{1,2,N}(\mathbf{P}, \mathbf{Q}), \begin{vmatrix} \vdots \\ \vdots \end{vmatrix} \Psi_{1,3,4}(\mathbf{P}, \mathbf{Q}), \begin{vmatrix} \vdots \\ \vdots \end{vmatrix} \cdots \begin{vmatrix} \vdots \\ \vdots \end{vmatrix} \Psi_{1,3,N}(\mathbf{P}, \mathbf{Q}), \begin{vmatrix} \vdots \\ \vdots \end{vmatrix} \cdots \begin{vmatrix} \vdots \\ \vdots \end{vmatrix} \Psi_{1,N-1,N}(\mathbf{P}, \mathbf{Q}))$

$\in \mathbb{R}^{(N-1)(N-2)}$ is a constraint function vector. Thus, the $(N-1)$ vectors $\mathbf{P}_{k,k+1}$

($1 \leq k \leq N-1$) are a basis for the manifold \mathcal{P} . Each vector $\mathbf{P}_{ij} \in \mathbb{R}^{(N-1)(N-2)}$

($1 \leq i < i+2 \leq j \leq N$) related to a non-chained momentum vector \mathbf{p}_{ij} can be uniquely expressed as follows:

$$\frac{\mathbf{P}_{ij} \mathbf{L}(\mathbf{Q}_{ij})^\top}{|\mathbf{Q}_{ij}|^2} = m_i m_j \sum_{k=i}^{j-1} \frac{1}{m_k m_{k+1}} \frac{\mathbf{P}_{k,k+1} \mathbf{L}(\mathbf{Q}_{k,k+1})^\top}{|\mathbf{Q}_{k,k+1}|^2}, \quad 1 \leq i < i+2 \leq j \leq N. \quad (25)$$

We solve equation (25) for \mathbf{P}_{ij} ($1 \leq i < i+2 \leq j \leq N$) and subsequently substitute equation (23) into the resulting solution. Then, we obtain

$$\begin{aligned} \mathbf{P}_{ij} &= \tilde{\mathbf{P}}_{ij} \\ &\equiv m_i m_j \left[\begin{array}{c} \tilde{Q}_{i,j[1]} \sum_{k=i}^{j-1} \frac{P_{k,k+1[1]} Q_{k,k+1[1]} - P_{k,k+1[2]} Q_{k,k+1[2]}}{m_k m_{k+1} |\mathbf{Q}_{k,k+1}|^2} + \tilde{Q}_{i,j[2]} \sum_{k=i}^{j-1} \frac{P_{k,k+1[1]} Q_{k,k+1[2]} + P_{k,k+1[2]} Q_{k,k+1[1]}}{m_k m_{k+1} |\mathbf{Q}_{k,k+1}|^2} \\ \tilde{Q}_{i,j[1]} \sum_{k=i}^{j-1} \frac{P_{k,k+1[1]} Q_{k,k+1[2]} + P_{k,k+1[2]} Q_{k,k+1[1]}}{m_k m_{k+1} |\mathbf{Q}_{k,k+1}|^2} - \tilde{Q}_{i,j[2]} \sum_{k=i}^{j-1} \frac{P_{k,k+1[1]} Q_{k,k+1[1]} - P_{k,k+1[2]} Q_{k,k+1[2]}}{m_k m_{k+1} |\mathbf{Q}_{k,k+1}|^2} \end{array} \right]^\top, \\ &\quad 1 \leq i < i+2 \leq j \leq N. \end{aligned} \quad (26)$$

Using equations (23) and (26), we can rewrite equation (19) as

$$\begin{aligned} \sum_{i=1}^{j-2} \tilde{\mathbf{G}}_{ij} - \sum_{i=j+2}^N \tilde{\mathbf{G}}_{ji} &= -\mathbf{G}_{j-1,j} + \mathbf{G}_{j,j+1}, \quad 2 \leq j \leq N-1, \\ \sum_{i=1}^{N-2} \tilde{\mathbf{G}}_{i,N} &= -\mathbf{G}_{N-1,N}, \end{aligned} \quad (27)$$

where

$$\tilde{\mathbf{G}}_{ij} \equiv \frac{1}{2|\tilde{\mathbf{Q}}_{ij}|^2} \left(\frac{d}{dt} \tilde{\mathbf{P}}_{ij} - \left(\frac{m}{4m_i m_j} |\tilde{\mathbf{P}}_{ij}|^2 - 2m_i m_j \right) \frac{\tilde{\mathbf{Q}}_{ij}}{|\tilde{\mathbf{Q}}_{ij}|^4} \right) \mathbf{L}^\top(\tilde{\mathbf{Q}}_{ij}), \quad 1 \leq i < i+2 \leq j \leq N,$$

and $\mathbf{G}_{j-1,j}$, $\mathbf{G}_{j,j+1}$, and $\mathbf{G}_{N-1,N}$ are defined by equation (16). Note that equation (27) is described by only $4(N-1)$ variables, $Q_{k,k+1[1]}$, $Q_{k,k+1[2]}$, $P_{k,k+1[1]}$, and $P_{k,k+1[2]}$, related to the chained vectors $\mathbf{q}_{k,k+1}$ and $\mathbf{p}_{k,k+1}$ ($1 \leq k \leq N-1$). In addition, the $2(N-1)$ vectors $\mathbf{Q}_{k,k+1}$ and $\mathbf{P}_{k,k+1}$ ($1 \leq k \leq N-1$) satisfy equation (15a); namely,

$$\frac{d}{dt} \mathbf{Q}_{k,k+1} = \frac{m}{4m_k m_{k+1}} \frac{\mathbf{P}_{k,k+1}}{|\mathbf{Q}_{k,k+1}|^2}, \quad 1 \leq k \leq N-1. \quad (28)$$

We call the system composed of equations (27) and (28) the *chain regularization of GNB* (*CRGNBP*). We clarify that the CRGNBP describes the same motion as the RGNBP composed of equations (15a), (17), and (19) in the following lemma.

Lemma 1 *Suppose*

(i) The $2(N - 1)$ vectors $\mathbf{Q}_{k,k+1}$ and $\mathbf{P}_{k,k+1}$ ($1 \leq k \leq N - 1$) are the solutions of the CRGNBP composed of equations (27) and (28).

(ii) The $(N - 1)(N - 2)$ vectors \mathbf{Q}_{ij} and \mathbf{P}_{ij} ($1 \leq i < i + 2 \leq j \leq N$) are given by equations (23) and (26).

Then, the $N(N - 1)$ vectors \mathbf{Q}_{ij} and \mathbf{P}_{ij} ($1 \leq i < j \leq N$) satisfy the RGNBP composed of equations (15a), (17), and (19).

Proof.

(a) *Derivation of equation (15a)*

The vectors \mathbf{Q}_{ij} and \mathbf{P}_{ij} ($1 \leq i < i + 2 \leq j \leq N$) defined by equations (23) and (26) satisfy (25). In addition, using equation (28), equation (25) is rewritten as

$$\frac{m}{4m_1m_j} \frac{\mathbf{P}_{ij}\mathbf{L}(\mathbf{Q}_{ij})^\top}{|\mathbf{Q}_{ij}|^2} = \sum_{k=i}^{j-1} \frac{d}{dt} \mathbf{Q}_{k,k+1} \mathbf{L}(\mathbf{Q}_{k,k+1})^\top, \quad 1 \leq i < i + 2 \leq j \leq N. \quad (29)$$

Similarly, the vectors \mathbf{Q}_{ij} ($1 \leq i < i + 2 \leq j \leq N$) in equation (23) satisfy equation (22), so the time derivative of (22) is also fulfilled. The derivative is written in the following form:

$$\frac{d}{dt} \mathbf{Q}_{ij} \mathbf{L}(\mathbf{Q}_{ij})^\top = \sum_{k=i}^{j-1} \frac{d}{dt} \mathbf{Q}_{k,k+1} \mathbf{L}(\mathbf{Q}_{k,k+1})^\top, \quad 1 \leq i < i + 2 \leq j \leq N. \quad (30)$$

Because the r.h.s. of equation (29) coincides with the r.h.s. of equation (30), the left-hand sides of equations (29) and (30) are equal. Accordingly, equation (15a) in the CRGNBP is obtained.

(b) *Derivation of equation (17)*

We have already shown that equation (23) is equivalent to equation (22). In addition, substituting equation (22) into $\Phi_{1jk}(\mathbf{Q})$ in equation (17) yields $\mathbf{0}$. Accordingly, the vectors \mathbf{Q}_{ij} ($1 \leq i < i + 2 \leq j \leq N$) in equation (23) satisfy equation (17).

(c) *Derivation of equation (19)*

Substituting equations (23) and (26) into equation (19) yields equation (27). Namely, equation (27) is equal to equation (19) under the conditions in equations (23) and (26). Thus, the $N(N - 1)$ vectors \mathbf{Q}_{ij} and \mathbf{P}_{ij} ($1 \leq i < j \leq N$) satisfy equation (19). \square

For large N , the number of dependent variables $P_{i,i+1[1]}$, $P_{i,i+1[2]}$, $Q_{i,i+1[1]}$, and $Q_{i,i+1[2]}$ in the CRGNBP, $4(N - 1)$, is remarkably smaller than $2N(N - 1)$, which is the number of dependent variables in the RGNBP composed of equations (15a), (17), and (19). Therefore, the computational cost of the CRGNBP is much lower than that of the RGNBP.

3. Energy-momentum Integrator for General N -body Problem

We use the d'Alembert-type scheme (Betsch 2005) to discretize the GNBPs, which leads to the G3BP for $N = 3$. Regardless of the number of masses N , the forms of equation (15a) and the functions \mathbf{G}_{ij} defined by equation (16) are invariant for each (i, j) .

In Section 3.1, we discretize the RGNBP composed of equations (15a), (17), and (19) so that, for $N = 3$, the discretized forms of equation (15a) and $\mathbf{G}_{i,j}$ coincide with the counterparts of the d'Alembert-type scheme for the G3BP (Minesaki 2013a). However, because the discrete-time problem including these forms has redundant dependent variables, it suffers from high computational cost for a large number of masses N . In Section 3.2, we remove the redundancy using some constraints, so we obtain the discretization of the CRGNBP composed of equations (27) and (28). The resulting discrete-time problem preserves all the conserved quantities except for the angular momentum precisely.

3.1. Discrete-time General N -body Problem with Redundant Variables

For $N = 3$, the RGNBP composed of equations (15a), (17), and (19) represents the regularized G3BP:

$$\begin{cases} \frac{d}{dt}\mathbf{Q}_{ij} = \frac{m}{4m_im_j} \frac{\mathbf{P}_{ij}}{|\mathbf{Q}_{ij}|^2}, \quad 1 \leq i < j \leq 3, \\ \mathbf{G}_{12} - \mathbf{G}_{23} = \mathbf{0}, \quad \mathbf{G}_{13} + \mathbf{G}_{23} = \mathbf{0}, \\ \Phi_{123}(\mathbf{Q}) = \mathbf{0}, \end{cases} \quad (31)$$

where \mathbf{G}_{ij} is defined by equation (16). Applying the extension of the d'Alembert-type scheme in (Betsch 2005), we gave the d-G3BP (Minesaki 2013a) with a time step $\Delta t = t^{(n+1)} - t^{(n)}$, namely, the discretization of system (31)²:

$$\begin{cases} \mathbf{F}_{ij}^{(n+1)} = \mathbf{0}, \quad 1 \leq i < j \leq 3, \\ \mathbf{G}_{12}^{(n+1)} - \mathbf{G}_{23}^{(n+1)} = \mathbf{0}, \quad \mathbf{G}_{13}^{(n+1)} + \mathbf{G}_{23}^{(n+1)} = \mathbf{0}, \\ \Phi_{123}(\mathbf{Q}^{(n+1)}) = \mathbf{0}, \end{cases} \quad (32)$$

where $\mathbf{Q}_{ij}^{(l)} = (Q_{ij[1]}^{(l)}, Q_{ij[2]}^{(l)})$ and $\mathbf{P}_{ij}^{(l)} = (P_{ij[1]}^{(l)}, P_{ij[2]}^{(l)})$ at time $t^{(l)}$ ($l = n, n+1$) for $1 \leq i < j \leq 3$; we define the midpoint value $(\bullet)^{(n+1/2)} \equiv [(\bullet)^{(n+1)} + (\bullet)^{(n)}] / 2$ of the function $(\bullet)(t)$, and

$$\mathbf{F}_{ij}^{(n+1)} \equiv \frac{\mathbf{Q}_{ij}^{(n+1)} - \mathbf{Q}_{ij}^{(n)}}{\Delta t} - \frac{m}{8m_im_j} \frac{|\mathbf{Q}_{ij}^{(n+1)}|^2 + |\mathbf{Q}_{ij}^{(n)}|^2}{|\mathbf{Q}_{ij}^{(n+1)}|^2 |\mathbf{Q}_{ij}^{(n)}|^2} \mathbf{P}_{ij}^{(n+1/2)}, \quad 1 \leq i < j \leq 3, \quad (33)$$

$$\begin{aligned} & \mathbf{G}_{ij}^{(n+1)} \\ & \equiv \frac{1}{2|\mathbf{Q}_{ij}^{(n+1/2)}|^2} \left(\frac{\mathbf{P}_{ij}^{(n+1)} - \mathbf{P}_{ij}^{(n)}}{\Delta t} - \frac{1}{|\mathbf{Q}_{ij}^{(n+1)}|^2 |\mathbf{Q}_{ij}^{(n)}|^2} \left(\frac{m}{8m_im_j} (|\mathbf{P}_{ij}^{(n+1)}|^2 + |\mathbf{P}_{ij}^{(n)}|^2) - 2m_im_j \right) \mathbf{Q}_{ij}^{(n+1/2)} \right) \mathbf{L} \left(\mathbf{Q}_{ij}^{(n+1/2)} \right)^\top, \\ & \quad \quad \quad 1 \leq i < j \leq 3. \end{aligned} \quad (34)$$

² This discretization coincides with discrete-time system (43) in (Minesaki 2013a). Note that the vectors $(\bullet)_{12}$, $(\bullet)_{23}$, and $(\bullet)_{13}$ correspond to $(\bullet)_3$, $(\bullet)_1$, and $-(\bullet)_2$ in (Minesaki 2013a), respectively.

For $N \geq 3$, the RGNBP composed of equations (15a), (17), and (19) involves the same \mathbf{F}_{ij} and \mathbf{G}_{ij} as the d-G3BP in equation (32) outside the range of i and j . Therefore, ignoring this range, we adopt $\mathbf{F}_{ij}^{(n+1)}$ and $\mathbf{G}_{ij}^{(n+1)}$ defined by equations (33) and (34) as the discrete analogs of \mathbf{F}_{ij} and \mathbf{G}_{ij} . Concretely, the discrete-time system, which approximately describes the motion of the RGNBP in a typical time interval $I^{(n)} = [t^{(n)}, t^{(n+1)}]$ with a corresponding time step $\Delta t = t^{(n+1)} - t^{(n)}$, is given as follows:

$$\begin{cases} \mathbf{F}_{ij}^{(n+1)} = \mathbf{0}, & 1 \leq i < j \leq N, \end{cases} \quad (35a)$$

$$\begin{cases} \sum_{i=1}^{j-1} \mathbf{G}_{ij}^{(n+1)} - \sum_{i=j+1}^N \mathbf{G}_{ji}^{(n+1)} = \mathbf{0}, & 2 \leq j \leq N, \end{cases} \quad (35b)$$

$$\begin{cases} \Phi_{1ij}(\mathbf{Q}^{(n+1)}) = \mathbf{0}, & 2 \leq i < j \leq N. \end{cases} \quad (35c)$$

$\mathbf{Q}^{(n)} \in \mathcal{Q}$ and $\mathbf{P}^{(n)} \in \mathbb{R}^{N(N-1)}$ are given quantities at time node $t^{(n)}$, where \mathcal{Q} was already defined by (21). In the following, we call this system the *discrete-time redundant general N -body problem (d-RGNBP)*. It can be used to calculate the unknown $\mathbf{Q}^{(n+1)}$ and $\mathbf{P}^{(n+1)} = (\mathbf{P}_{1,2}^{(n+1)}, \begin{vmatrix} \vdots \\ \vdots \end{vmatrix} \dots \begin{vmatrix} \vdots \\ \vdots \end{vmatrix} \mathbf{P}_{1,N}^{(n+1)}, \left| \mathbf{P}_{2,3}^{(n+1)}, \begin{vmatrix} \vdots \\ \vdots \end{vmatrix} \dots \begin{vmatrix} \vdots \\ \vdots \end{vmatrix} \mathbf{P}_{2,N}^{(n+1)}, \right| \dots \left| \mathbf{P}_{N-1,N}^{(n+1)} \right|) \in \mathbb{R}^{N(N-1)}$. Note that $\mathbf{Q}^{(n+1)} \in \mathcal{Q}$ because of equation (35c).

The d-RGNBP preserves all the conserved quantities, except the angular momentum, precisely. This is shown in the following theorem.

Theorem 1 (*Conserved quantities of d-RGNBP*) *The d-RGNBP (35) keeps the following three conserved quantities exactly:*

1. the Hamiltonian defined by (1),

2. the linear momentum $\mathbf{l} \equiv \sum_{i=1}^N \mathbf{p}'_i = \mathbf{0}$,

3. the position of the center of mass $\mathbf{c} \equiv \sum_{i=1}^N m_i \mathbf{q}'_i = \mathbf{0}$.

Proof.

1. By using the multipliers $\Lambda_{ij} = (\Lambda_{ij[1]}, \Lambda_{ij[2]})$ ($2 \leq i < j \leq N$), the d-RGNBP (35) is rewritten as

$$\begin{cases} \mathbf{F}_{ij}^{(n+1)} = \mathbf{0}, & 1 \leq i < j \leq N, \end{cases} \quad (36a)$$

$$\begin{cases} \mathbf{G}_{1j}^{(n+1)} - \sum_{k=2}^{j-1} \Lambda_{kj} + \sum_{k=j+1}^N \Lambda_{jk} = \mathbf{0}, & j = 2, \dots, N, \end{cases} \quad (36b)$$

$$\begin{cases} \mathbf{G}_{ij}^{(n+1)} + \Lambda_{ij} = \mathbf{0}, & 2 \leq i < j \leq N. \end{cases} \quad (36c)$$

By using equation (36c), equation (36b) is expressed as equation (35b).

We set the following functions:

$$H_{1j}(\mathbf{P}_{1j}, \mathbf{Q}_{1j}) \equiv \frac{m}{8m_1m_j} \frac{|\mathbf{P}_{1j}|^2}{|\mathbf{Q}_{1j}|^2} - \frac{m_1m_j}{|\mathbf{Q}_{1j}|^2} + \mathbf{Q}_{1j} \mathbf{L}(\mathbf{Q}_{1j})^\top \cdot \left(\sum_{i=2}^{j-1} \Lambda_{ij} - \sum_{i=j+1}^N \Lambda_{ji} \right), \quad 2 \leq j \leq N,$$

$$H_{ij}(\mathbf{P}_{ij}, \mathbf{Q}_{ij}) \equiv \frac{m}{8m_im_j} \frac{|\mathbf{P}_{ij}|^2}{|\mathbf{Q}_{ij}|^2} - \frac{m_im_j}{|\mathbf{Q}_{ij}|^2} + \mathbf{Q}_{ij} \mathbf{L}(\mathbf{Q}_{ij})^\top \cdot \Lambda_{ij}, \quad 2 \leq i < j \leq N.$$

Because (17) and

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^N H_{ij}(\mathbf{P}_{ij}, \mathbf{Q}_{ij}) = \sum_{i=1}^{N-1} \sum_{j=i+1}^N \left(\frac{m}{8m_im_j} \frac{|\mathbf{P}_{ij}|^2}{|\mathbf{Q}_{ij}|^2} - \frac{m_im_j}{|\mathbf{Q}_{ij}|^2} \right) + \sum_{j=2}^{N-1} \sum_{k=j+1}^N \Phi_{1jk}(\mathbf{Q}) \cdot \Lambda_{jk},$$

the value of the r.h.s. coincides with h_{LC} in equation (20), which equals the value of H_{LC} defined by equation (18). For $2 \leq j \leq N$, scalar multiplication of equation (36a) for $i = 1$ and equation (36b) by $\mathbf{P}_{1j}^{(n+1)} - \mathbf{P}_{1j}^{(n)}$ and $-(\mathbf{Q}_{1j}^{(n+1)} - \mathbf{Q}_{1j}^{(n)})$, respectively, and subsequent addition of the two equations yield

$$\begin{aligned} & \frac{m}{8m_1m_j} \frac{|\mathbf{P}_{1j}^{(n+1)}|^2}{|\mathbf{Q}_{1j}^{(n+1)}|^2} - \frac{m}{8m_1m_j} \frac{|\mathbf{P}_{1j}^{(n)}|^2}{|\mathbf{Q}_{1j}^{(n)}|^2} - \left(\frac{m_1m_j}{|\mathbf{Q}_{1j}^{(n+1)}|^2} - \frac{m_1m_j}{|\mathbf{Q}_{1j}^{(n)}|^2} \right) \\ & + \left(\mathbf{Q}_{1j}^{(n+1)} \mathbf{L}(\mathbf{Q}_{1j}^{(n+1)})^\top - \mathbf{Q}_{1j}^{(n)} \mathbf{L}(\mathbf{Q}_{1j}^{(n)})^\top \right) \cdot \left(\sum_{i=2}^{j-1} \Lambda_{ij} - \sum_{i=j+1}^N \Lambda_{ji} \right) = \mathbf{0}, \quad 2 \leq j \leq N. \end{aligned} \quad (37)$$

Equation (37) represents the conservation of $H_{1j}(\mathbf{P}_{1j}, \mathbf{Q}_{1j})$ because it equals

$H_{1j}(\mathbf{P}_{1j}^{(n+1)}, \mathbf{Q}_{1j}^{(n+1)}) - H_{1j}(\mathbf{P}_{1j}^{(n)}, \mathbf{Q}_{1j}^{(n)}) = 0$. In addition, for $1 < i < j \leq N$, scalar multiplication of equation (36a) for $i \geq 2$ and equation (36c) by $\mathbf{P}_{ij}^{(n+1)} - \mathbf{P}_{ij}^{(n)}$ and

$-(\mathbf{Q}_{ij}^{(n+1)} - \mathbf{Q}_{ij}^{(n)})$, respectively, and subsequent addition of the two equations lead to

$$\begin{aligned} & \frac{m}{8m_i m_j} \frac{|\mathbf{P}_{ij}^{(n+1)}|^2}{|\mathbf{Q}_{ij}^{(n+1)}|^2} - \frac{m}{8m_i m_j} \frac{|\mathbf{P}_{ij}^{(n)}|^2}{|\mathbf{Q}_{ij}^{(n)}|^2} - \left(\frac{m_i m_j}{|\mathbf{Q}_{ij}^{(n+1)}|^2} - \frac{m_i m_j}{|\mathbf{Q}_{ij}^{(n)}|^2} \right) \\ & + \left(\mathbf{Q}_{ij}^{(n+1)} \mathbf{L}(\mathbf{Q}_{ij}^{(n+1)})^\top - \mathbf{Q}_{ij}^{(n)} \mathbf{L}(\mathbf{Q}_{ij}^{(n)})^\top \right) \cdot \boldsymbol{\Lambda}_{ij} = \mathbf{0}, \quad 1 < i < j \leq N. \end{aligned} \quad (38)$$

Equation (38) describes the conservation of $H_{ij}(\mathbf{P}_{ij}, \mathbf{Q}_{ij})$ because it is equivalent to $H_{ij}(\mathbf{P}_{ij}^{(n+1)}, \mathbf{Q}_{ij}^{(n+1)}) - H_{ij}(\mathbf{P}_{ij}^{(n)}, \mathbf{Q}_{ij}^{(n)}) = 0$. Accordingly, the d-RGNBP (35) preserves the value of h_{LC} defined by equation (20). The d-RGNBP (35) conserves the value of H defined by equation (1) because the value h_{LC} is that of H_{LC} in equation (18), and the value of h_{LC} equals that of H in equation (1), as described in Section 2.2.2.

2 . The linear momentum $\mathbf{l}^{(n+1)}$ in the barycentric frame at time node $t^{(n+1)}$ is given by

$$\mathbf{l}^{(n+1)} = \sum_{i=1}^N \mathbf{p}_i'^{(n+1)}. \quad (39)$$

Substituting equation (6) into equation (39), we see

$$\mathbf{l}^{(n+1)} = \mathbf{0}.$$

Therefore, the d-GNBP (35) preserves the linear momentum \mathbf{l} , which is identically zero.

3. The position of the center of mass $\mathbf{c}^{(n+1)}$ in the barycentric frame at time node $t^{(n+1)}$ is expressed by

$$\mathbf{c}^{(n+1)} = \sum_{i=1}^N m_i \mathbf{q}_i'^{(n+1)}. \quad (40)$$

We substitute equation (6) into equation (40). Then, we can check

$$\mathbf{c}^{(n+1)} = \mathbf{0}.$$

As a result, the d-RGNBP (35) keeps the vector value of the position of the center of motion \mathbf{c} at zero. \square

For large N , the number of variables $P_{ij[1]}^{(n+1)}$, $P_{ij[2]}^{(n+1)}$, $Q_{ij[1]}^{(n+1)}$, and $Q_{ij[1]}^{(n+1)}$ in the d-RGNBP, $2N(N-1)$, is remarkably larger than $4N$, which is the number of dependent variables $p'_{i[1]}$, $p'_{i[2]}$, $q'_{i[1]}$, and $q'_{i[2]}$ in the GNPB (2). Thus, the computational cost of the d-RGNBP is very high.

3.2. Discrete-time General N -body Problem with Chain Variables

In this section, to reduce the high computational cost of the d-RGNBP, we rewrite the d-RGNBP using only $\mathbf{Q}_{k,k+1}^{(n)}$, $\mathbf{Q}_{k,k+1}^{(n+1)}$, $\mathbf{P}_{k,k+1}^{(n)}$, and $\mathbf{P}_{k,k+1}^{(n+1)}$ ($1 \leq k \leq N-1$) related to chained vectors.

First, we show that the vectors $\mathbf{Q}_{ij}^{(l)}$ ($1 \leq i < i+2 \leq j \leq N$, $l = n, n+1$) can be expressed as the functions of $\mathbf{Q}_{k,k+1}^{(l)}$ ($1 \leq k \leq N-1$, $l = n, n+1$) related to the chain positional vectors $\mathbf{q}_{k,k+1}^{(l)}$ ($1 \leq k \leq N-1$, $l = n, n+1$). Because $\mathbf{Q}^{(l)} \in \mathcal{Q}$ ($l = n, n+1$), where \mathcal{Q} is a $2(N-1)$ -dimensional manifold defined by equation (21), we can assume that the $(N-1)$ vectors $\mathbf{Q}_{k,k+1}^{(l)}$ ($1 \leq k \leq N-1$, $l = n, n+1$), which correspond to $2(N-1)$ variables, constitute a basis for the manifold \mathcal{Q} . In this basis, each vector $\mathbf{Q}_{ij}^{(l)} \in \mathcal{Q}$ ($1 \leq i < i+2 \leq j \leq N$, $l = n, n+1$) related to a non-chained position vector $\mathbf{q}_{ij}^{(l)}$ ($l = n, n+1$) satisfies

$$\mathbf{Q}_{ij}^{(n+1)} \mathbf{L}(\mathbf{Q}_{ij}^{(n+1)})^\top = \sum_{k=i}^{j-1} \mathbf{Q}_{k,k+1}^{(n+1)} \mathbf{L}(\mathbf{Q}_{k,k+1}^{(n+1)})^\top, \quad 1 \leq i < i+2 \leq j \leq N. \quad (41)$$

In addition, to avoid cancellation of significant digits, the solutions of equation (41) for $\mathbf{Q}_{ij}^{(l)}$ ($l = n, n+1$) are written as follows:

$$\mathbf{Q}_{ij}^{(l)} = \tilde{\mathbf{Q}}_{ij}^{(l)}$$

$$\equiv \left\{ \begin{array}{l} \left(\frac{\sqrt{2} \sum_{k=i}^{j-1} Q_{k,k+1[1]}^{(l)} Q_{k,k+1[2]}^{(l)}}{\sqrt{a_{ij}^{(l)} - b_{ij}^{(l)}}}, \sqrt{\frac{a_{ij}^{(l)} - b_{ij}^{(l)}}{2}} \right) \text{ if } b_{ij}^{(l)} < 0, \\ \left(\sqrt{\frac{a_{ij}^{(l)} + b_{ij}^{(l)}}{2}}, \frac{\sqrt{2} \sum_{k=i}^{j-1} Q_{k,k+1[1]}^{(l)} Q_{k,k+1[2]}^{(l)}}{\sqrt{a_{ij}^{(l)} + b_{ij}^{(l)}}} \right) \text{ if } b_{ij}^{(l)} \geq 0, \ 1 \leq i < i+2 \leq j \leq N, \ l = n, n+1, \end{array} \right. \quad (42)$$

where

$$a_{ij}^{(l)} \equiv \sqrt{\left(\sum_{k=i}^{j-1} \left((Q_{k,k+1[1]}^{(l)})^2 - (Q_{k,k+1[2]}^{(l)})^2 \right) \right)^2 + 4 \left(\sum_{k=i}^{j-1} Q_{k,k+1[1]}^{(l)} Q_{k,k+1[2]}^{(l)} \right)^2},$$

$$b_{ij}^{(l)} \equiv \sum_{k=i}^{j-1} \left((Q_{k,k+1[1]}^{(l)})^2 - (Q_{k,k+1[2]}^{(l)})^2 \right), \quad 1 \leq i < i+2 \leq j \leq N, \ l = n, n+1.$$

Next, we write the vectors $\mathbf{P}_{ij}^{(n+1/2)}$ ($1 \leq i < i+2 \leq j \leq N$) related to the non-chained momentum vectors $\mathbf{p}_{ij}^{(n+1/2)}$ as functions of $\mathbf{P}_{k,k+1}^{(n+1/2)}$, $\mathbf{Q}_{k,k+1}^{(n)}$, and $\mathbf{Q}_{k,k+1}^{(n+1)}$ ($1 \leq k \leq N-1$). $\mathbf{Q}_{ij}^{(n+1)} \in \mathcal{Q}$ ($1 \leq i < i+2 \leq j \leq N$) are expressed by equation (42) using the basis constituted by the $(N-1)$ vectors $\mathbf{Q}_{k,k+1}^{(n+1)}$ ($1 \leq k \leq N-1$). As described in Section 3.1, $\mathbf{Q}^{(n)} \in \mathcal{Q}$ is also satisfied; namely, $\mathbf{Q}^{(n)}$ is restricted to the constraints $\Phi(\mathbf{Q}^{(n)}) = \mathbf{0}_{1 \times (N-1)(N-2)}$. The form of these constraints is the same as that of the constraints $\Phi(\mathbf{Q}^{(n+1)}) = \mathbf{0}_{1 \times (N-1)(N-2)}$, so $\mathbf{Q}^{(n)}$ satisfies the same form in (41):

$$\mathbf{Q}_{ij}^{(n)} \mathbf{L}(\mathbf{Q}_{ij}^{(n)})^\top = \sum_{k=i}^{j-1} \mathbf{Q}_{k,k+1}^{(n)} \mathbf{L}(\mathbf{Q}_{k,k+1}^{(n)})^\top, \quad 1 \leq i < i+2 \leq j \leq N. \quad (43)$$

Subtraction of equation (43) from equation (41) and subsequent division by $2\Delta t = 2(t^{(n+1)} - t^{(n)})$ yield

$$\frac{1}{2\Delta t} \left(\mathbf{Q}_{ij}^{(n+1)} \mathbf{L}(\mathbf{Q}_{ij}^{(n+1)})^\top - \mathbf{Q}_{ij}^{(n)} \mathbf{L}(\mathbf{Q}_{ij}^{(n)})^\top \right) = \sum_{k=i}^{j-1} \frac{1}{2\Delta t} \left(\mathbf{Q}_{k,k+1}^{(n+1)} \mathbf{L}(\mathbf{Q}_{k,k+1}^{(n+1)})^\top - \mathbf{Q}_{k,k+1}^{(n)} \mathbf{L}(\mathbf{Q}_{k,k+1}^{(n)})^\top \right),$$

$$1 \leq i < i+2 \leq j \leq N.$$

This leads to

$$\frac{\mathbf{Q}_{ij}^{(n+1)} - \mathbf{Q}_{ij}^{(n)}}{\Delta t} \mathbf{L}(\mathbf{Q}_{ij}^{(n+1/2)})^\top = \sum_{k=i}^{j-1} \frac{\mathbf{Q}_{k,k+1}^{(n+1)} - \mathbf{Q}_{k,k+1}^{(n)}}{\Delta t} \mathbf{L}(\mathbf{Q}_{k,k+1}^{(n+1/2)})^\top, \quad 1 \leq i < i+2 \leq j \leq N. \quad (44)$$

By using equations (35a) and (42), equation (44) is rewritten as

$$\frac{m}{8m_i m_j} \frac{|\tilde{\mathbf{Q}}_{ij}^{(n+1)}|^2 + |\tilde{\mathbf{Q}}_{ij}^{(n)}|^2}{|\tilde{\mathbf{Q}}_{ij}^{(n+1/2)}|^2 |\tilde{\mathbf{Q}}_{ij}^{(n)}|^2} \mathbf{P}_{ij}^{(n+1/2)} \mathbf{L}(\tilde{\mathbf{Q}}_{ij}^{(n+1/2)})^\top = \sum_{k=i}^{j-1} \frac{m}{8m_k m_{k+1}} \frac{|\mathbf{Q}_{k,k+1}^{(n+1)}|^2 + |\mathbf{Q}_{k,k+1}^{(n)}|^2}{|\mathbf{Q}_{k,k+1}^{(n+1/2)}|^2 |\mathbf{Q}_{k,k+1}^{(n)}|^2} \mathbf{P}_{k,k+1}^{(n+1/2)} \mathbf{L}(\mathbf{Q}_{k,k+1}^{(n+1/2)})^\top, \quad 1 \leq i < i+2 \leq j \leq N. \quad (45)$$

We solve equation (45) for $\mathbf{P}_{ij}^{(n+1/2)}$ ($1 \leq i < i+2 \leq j \leq N$) and subsequently substitute equation (42) into the resulting solution. Then, we give

$$\mathbf{P}_{ij}^{(n+1/2)} = \tilde{\mathbf{P}}_{ij}^{(n+1/2)} \equiv \left(\tilde{P}_{ij[1]}^{(n+1/2)}, \tilde{P}_{ij[2]}^{(n+1/2)} \right), \quad 1 \leq i < i+2 \leq j \leq N, \quad (46)$$

where

$$\begin{aligned} \tilde{P}_{ij[1]}^{(n+1/2)} &= m_i m_j \frac{|\tilde{\mathbf{Q}}_{ij}^{(n+1)}|^2 |\tilde{\mathbf{Q}}_{ij}^{(n)}|^2}{|\tilde{\mathbf{Q}}_{ij}^{(n+1/2)}|^2 \left(|\tilde{\mathbf{Q}}_{ij}^{(n+1)}|^2 + |\tilde{\mathbf{Q}}_{ij}^{(n)}|^2 \right)} \\ &\quad \left(\tilde{Q}_{ij[1]}^{(n+1/2)} \sum_{k=i}^{j-1} \frac{\left(|\mathbf{Q}_{k,k+1}^{(n+1)}|^2 + |\mathbf{Q}_{k,k+1}^{(n)}|^2 \right) \left(P_{k,k+1[1]}^{(n+1/2)} Q_{k,k+1[1]}^{(n+1/2)} - P_{k,k+1[2]}^{(n+1/2)} Q_{k,k+1[2]}^{(n+1/2)} \right)}{m_k m_{k+1} |\mathbf{Q}_{k,k+1}^{(n+1/2)}|^2 |\mathbf{Q}_{k,k+1}^{(n)}|^2} \right. \\ &\quad \left. + \tilde{Q}_{ij[2]}^{(n+1/2)} \sum_{k=i}^{j-1} \frac{\left(|\mathbf{Q}_{k,k+1}^{(n+1)}|^2 + |\mathbf{Q}_{k,k+1}^{(n)}|^2 \right) \left(P_{k,k+1[1]}^{(n+1/2)} Q_{k,k+1[2]}^{(n+1/2)} + P_{k,k+1[2]}^{(n+1/2)} Q_{k,k+1[1]}^{(n+1/2)} \right)}{m_k m_{k+1} |\mathbf{Q}_{k,k+1}^{(n+1/2)}|^2 |\mathbf{Q}_{k,k+1}^{(n)}|^2} \right), \\ \tilde{P}_{ij[2]}^{(n+1/2)} &= m_i m_j \frac{|\tilde{\mathbf{Q}}_{ij}^{(n+1)}|^2 |\tilde{\mathbf{Q}}_{ij}^{(n)}|^2}{|\tilde{\mathbf{Q}}_{ij}^{(n+1/2)}|^2 \left(|\tilde{\mathbf{Q}}_{ij}^{(n+1)}|^2 + |\tilde{\mathbf{Q}}_{ij}^{(n)}|^2 \right)} \\ &\quad \left(\tilde{Q}_{ij[1]}^{(n+1/2)} \sum_{k=i}^{j-1} \frac{\left(|\mathbf{Q}_{k,k+1}^{(n+1)}|^2 + |\mathbf{Q}_{k,k+1}^{(n)}|^2 \right) \left(P_{k,k+1[1]}^{(n+1/2)} Q_{k,k+1[2]}^{(n+1/2)} + P_{k,k+1[2]}^{(n+1/2)} Q_{k,k+1[1]}^{(n+1/2)} \right)}{m_k m_{k+1} |\mathbf{Q}_{k,k+1}^{(n+1/2)}|^2 |\mathbf{Q}_{k,k+1}^{(n)}|^2} \right. \\ &\quad \left. - \tilde{Q}_{ij[2]}^{(n+1/2)} \sum_{k=i}^{j-1} \frac{\left(|\mathbf{Q}_{k,k+1}^{(n+1)}|^2 + |\mathbf{Q}_{k,k+1}^{(n)}|^2 \right) \left(P_{k,k+1[1]}^{(n+1/2)} Q_{k,k+1[1]}^{(n+1/2)} - P_{k,k+1[2]}^{(n+1/2)} Q_{k,k+1[2]}^{(n+1/2)} \right)}{m_k m_{k+1} |\mathbf{Q}_{k,k+1}^{(n+1/2)}|^2 |\mathbf{Q}_{k,k+1}^{(n)}|^2} \right), \end{aligned} \quad 1 \leq i < i+2 \leq j \leq N.$$

Because of the definition of the midpoint value $(\bullet)^{(n+1/2)}$ and equation (46),

$$\mathbf{P}_{ij}^{(n+1)} = \tilde{\mathbf{P}}_{ij}^{(n+1)} = 2\tilde{\mathbf{P}}_{ij}^{(n+1/2)} - \tilde{\mathbf{P}}_{ij}^{(n)} \quad (1 \leq i < j \leq N). \quad (47)$$

Using equations (42) and (46), we can rewrite equation (35b) in the d-RGNBP as

$$\sum_{i=1}^{j-2} \tilde{\mathbf{G}}_{ij}^{(n+1)} - \sum_{i=j+2}^N \tilde{\mathbf{G}}_{ji}^{(n+1)} = -\mathbf{G}_{j-1,j}^{(n+1)} + \mathbf{G}_{j,j+1}^{(n+1)}, \quad j = 2, \dots, N-1; \quad \sum_{i=1}^{N-2} \tilde{\mathbf{G}}_{i,N}^{(n+1)} = -\mathbf{G}_{N-1,N}^{(n+1)}, \quad (48)$$

where

$$\begin{aligned} & \tilde{\mathbf{G}}_{ij}^{(n+1)} \\ & \equiv \frac{1}{2|\tilde{\mathbf{Q}}_{ij}^{(n+1/2)}|^2} \left(\frac{\tilde{\mathbf{P}}_{ij}^{(n+1)} - \tilde{\mathbf{P}}_{ij}^{(n)}}{\Delta t} - \frac{1}{|\tilde{\mathbf{Q}}_{ij}^{(n+1)}|^2 |\tilde{\mathbf{Q}}_{ij}^{(n)}|^2} \left(\frac{m}{8m_i m_j} \left(|\tilde{\mathbf{P}}_{ij}^{(n+1)}|^2 + |\tilde{\mathbf{P}}_{ij}^{(n)}|^2 \right) - 2m_i m_j \right) \tilde{\mathbf{Q}}_{ij}^{(n+1/2)} \right) \mathbf{L} \left(\tilde{\mathbf{Q}}_{ij}^{(n+1/2)} \right)^\top, \\ & \quad 1 \leq i < i+2 \leq j \leq N, \end{aligned} \quad (49)$$

and $\mathbf{G}_{j-1,j}^{(n+1)}$, $\mathbf{G}_{j,j+1}^{(n+1)}$, and $\mathbf{G}_{N-1,N}^{(n+1)}$ have the same form as equation (34). Note that equation (49) is described by only $4(N-1)$ vectors, $\mathbf{Q}_{k,k+1}^{(n)}$, $\mathbf{Q}_{k,k+1}^{(n+1)}$, $\mathbf{P}_{k,k+1}^{(n)}$, and $\mathbf{P}_{k,k+1}^{(n+1)}$. In addition, these vectors satisfy equation (35a); namely,

$$\frac{\mathbf{Q}_{k,k+1}^{(n+1)} - \mathbf{Q}_{k,k+1}^{(n)}}{\Delta t} = \frac{m}{8m_k m_{k+1}} \frac{|\mathbf{Q}_{k,k+1}^{(n+1)}|^2 + |\mathbf{Q}_{k,k+1}^{(n)}|^2}{|\mathbf{Q}_{k,k+1}^{(n+1)}|^2 |\mathbf{Q}_{k,k+1}^{(n)}|^2} \mathbf{P}_{k,k+1}^{(n+1/2)}, \quad 1 \leq k \leq N-1. \quad (50)$$

We call the discrete-time system composed of equations (48) and (50) the *discrete-time chain regularization of the GGNBP (d-CRGNBP)*. The d-CRGNBP describes the same motion as the d-RGNBP (35), as shown by the following lemma.

Lemma 2 *Suppose*

- (i) *The $(N-1)(N-2)$ vectors $\mathbf{Q}_{ij}^{(n+1)}$ and $\mathbf{P}_{ij}^{(n+1)}$ ($1 \leq i < i+2 \leq j \leq N$) are given by equations (42) and (47).*
- (ii) *The $2(N-1)$ vectors $\mathbf{Q}_{k,k+1}^{(n+1)}$ and $\mathbf{P}_{k,k+1}^{(n+1)}$ ($1 \leq k \leq N-1$) are the solutions of the d-CRGNBP composed of equations (48) and (50).*

Then, the $N(N-1)$ vectors $\mathbf{Q}_{ij}^{(n+1)}$ and $\mathbf{P}_{ij}^{(n+1)}$ ($1 \leq i < j \leq N$) satisfy the d-RGNBP (35).

Proof.

- (a) *Derivation of equation (35a)*

The vectors $\mathbf{P}_{ij}^{(n+1/2)}$ ($1 \leq i < i+2 \leq j \leq N$) in equation (46) satisfy equation (45).

Further, through equation (50), equation (45) leads to

$$\frac{m}{8m_i m_j} \frac{|\mathbf{Q}_{ij}^{(n+1)}|^2 + |\mathbf{Q}_{ij}^{(n)}|^2}{|\mathbf{Q}_{ij}^{(n+1)}|^2 |\mathbf{Q}_{ij}^{(n)}|^2} \mathbf{P}_{ij}^{(n+1/2)} \mathbf{L}(\mathbf{Q}_{ij}^{(n+1/2)})^\top = \sum_{k=i}^{j-1} \frac{\mathbf{Q}_{k,k+1}^{(n+1)} - \mathbf{Q}_{k,k+1}^{(n)}}{\Delta t} \mathbf{L}(\mathbf{Q}_{k,k+1}^{(n+1/2)})^\top, \\ 1 \leq i < i+2 \leq j \leq N. \quad (51)$$

Similarly, the vectors $\mathbf{Q}_{ij}^{(n+1)}$ ($1 \leq i < i+2 \leq j \leq N$) defined by equation (42) fulfill equation (41) and $\mathbf{Q}_{ij}^{(n+1)} \in \mathcal{Q}$, so equation (44) is also fulfilled. Because the r.h.s. of equation (51) coincides with that of equation (44), equation (35a) in the d-RGNBP is given.

(b) *Derivation of equation (35b)*

Substitution of equations (42), (46), and (47) into equation (35b) yields equation (48).

Namely, equation (48), which is part of condition (ii), equals equation (35b) under condition (i). Accordingly, $\mathbf{Q}_{ij}^{(n+1)}$ and $\mathbf{P}_{ij}^{(n+1)}$ ($1 \leq i < j \leq N$) satisfying conditions (i) and (ii) follow from equation (35b).

(c) *Derivation of equation (35c)*

We have already stated that equation (42) is the solution of equation (41). Substitution of equation (41) into the l.h.s. of equation (35c) yields $\mathbf{0}$. Thus, $\mathbf{Q}_{ij}^{(n+1)}$ ($1 \leq i < j \leq N$) satisfy equation (35c) under conditions (i) and (ii). \square

Lemma 2 clarifies that the d-CRGNBP reproduces the motion of the d-RGNBP, and Theorem 1 shows that the d-RGNBP preserves all the conserved quantities but the angular momentum. Therefore, the conservation of quantities in the d-CRGNBP is stated in the following theorem.

Theorem 2 (*Conserved quantities of d-CRGNBP*) *The d-CRGNBP composed of equations (48) and (50) exactly preserves the following three conserved quantities:*

1. the Hamiltonian defined by equation (1),
2. the linear momentum $\mathbf{l} \equiv \sum_{i=1}^N \mathbf{p}'_i = \mathbf{0}$,
3. the position of the center of mass $\mathbf{c} \equiv \sum_{i=1}^N m_i \mathbf{q}'_i = \mathbf{0}$.

4. Numerical Results

In this section, we compare the results obtained with the following methods:

RK4: The fourth-order Runge–Kutta method, which is used for integrating equation (2),

SI4: The fourth-order symplectic method, which is applied to equation (2),

G: Greenspan’s energy conserving method (Greenspan 1974; LaBudde & Greenspan 1974), which is second-order accurate and is applied to equation (2),

d-CRGNBP: The d-CRGNBP, which is given by equations (48) and (50), and is second-order accurate.

In Section 4.1, we show that the d-CRGNBP precisely conserves the Hamiltonian of the general four-body problem (G4BP) for a long integration interval. Next, in Section 4.2, we show that the d-CRGNBP computes equilibrium solutions and periodic orbits around equilibrium points in the G3BP, G4BP, and general five-body problem (G5BP) more correctly than the other methods.

4.1. Conservation

First, let us show that the d-CRGNBP preserves the Hamiltonian H exactly and the angular momentum $\mathbf{j} = \sum_{i=1}^4 \mathbf{p}_i \times \mathbf{q}_i$ approximately. Figure 1 shows the dependence of the relative error growth of the Hamiltonian H and angular momentum \mathbf{j} for the G4BP on the RK4, SI4, G and d-CRGNBP methods, respectively. The adopted initial conditions are as follows:

$$\begin{aligned} m_1 &= m_2 = m_3 = m_4 = 0.25, \\ \mathbf{p}'_1 &= (0, 0.1), \quad \mathbf{p}'_2 = (0, -0.05), \quad \mathbf{p}'_3 = (0, -0.1), \quad \mathbf{p}'_4 = (0, 0.05), \\ \mathbf{q}'_1 &= (-10, 0), \quad \mathbf{q}'_2 = (-12, 0), \quad \mathbf{q}'_3 = (10, 0), \quad \mathbf{q}'_4 = (12, 0). \end{aligned} \tag{52}$$

In addition, the step size is fixed at $\Delta t = 0.1$. The initial condition corresponds to the Caledonian symmetric four-body problem, in which the four bodies are always configured in a parallelogram (Széll, Érdi, Sándor & Steves 2004). Each method always sets the linear momentum \mathbf{l} and the center of mass \mathbf{c} at the origin in the barycentric frame.

For the RK4 method, the relative error of H_{rel} grows with time t , whereas the error is bounded by a sufficiently small value 10^{-7} for the SI4, G, and d-CRGNBP methods. In particular, the G and d-CRGNBP methods conserve H with 10^{-15} accuracy. In addition, the error of \mathbf{j} grows in proportion to the time t for the RK4 method, whereas it is bounded by 10^{-5} for the SI4, G, and d-CRGNBP methods. Specifically, the SI4 and G methods precisely keep \mathbf{j} . Because only Greenspan’s energy conserving method (G) preserves both H and \mathbf{j} , it would appear that this method reproduces orbits more precisely than the others. However, Section 4.2 will clarify that this prediction is incorrect.

4.2. Periodic Orbits

In Section 4.1, we showed that the d-CRGNBP does not exactly preserve the angular momentum. Therefore, it is questionable whether the d-CRGNBP can reproduce various orbits of the G3BP and G4BP because the orbits lie on the manifold determined by conserved quantities. To answer this question, we show that various orbits computed by the d-CRGNBP accurately coincide with those of the G3BP, G4BP and G5BP.

4.2.1. Choreographies in the G3BP

In choreography solutions, all the bodies are equally spaced along a single closed orbit. The three-body figure-eight choreography was discovered by Chenciner and Montgomery (Chenciner 2000) and located numerically by Simó (Simó 2000). The initial conditions are those cited in (Simó 2000):

$$\begin{aligned}
 m_1 &= m_2 = m_3 = 1, \\
 \mathbf{p}'_1 &= (0.46620369, 0.43236573), \quad \mathbf{p}'_2 = (-0.93240737, -0.86473146), \\
 \mathbf{p}'_3 &= (0.46620369, 0.43236573), \quad \mathbf{q}'_1 = (0.97000436, -0.24308753), \\
 \mathbf{q}'_2 &= (0, 0), \quad \mathbf{q}'_3 = (-0.97000436, 0.24308753).
 \end{aligned} \tag{53}$$

Applying the RK4, SI4, G, and d-CRGNBP methods, we obtained the orbits of particle m_3 in the barycentric frame. We used the common time step $\Delta t = 0.1$ and integrated until $t_f = 10,000$ (see Figure 2). Particle m_3 theoretically follows a closed figure-eight orbit under this condition and travels more than 1500 times around the orbit. The SI4 and d-CRGNBP methods give the closed figure-eight orbit with high precision, whereas the RK4 and G methods do not obtain a closed orbit. In particular, the RK4 method shows the orbit of particle m_3 shifting away from the figure-eight orbit, and the G method obtains an orbit drifting around that orbit.

4.2.2. *Stable Equilibrium Points in the G4BP*

We clarify that the d-CRGNBP precisely computes some stable equilibrium solutions in the circular G4BP which has four finite masses m_1 , m_2 , m_3 and m_4 . As the mass m_4 goes to zero, the equilibrium solutions in the circular G4BP reduce to those in the circular restricted four-body problem (CR4BP), in each of which the three masses m_1 , m_2 and m_3 form an equilateral triangle and orbit a common circle according to the Lagrangian solution (Baltagiannis & Papadakis 2011a; Majorana 1981) in the barycentric inertial frame. The CR4BP has eight equilibrium points, at one of which the massless particle m_4 rests in a rotating frame where the three primaries, m_1 , m_2 , and m_3 are fixed at the vertices of an equilateral triangle. Two of the equilibrium points are linearly stable (Baltagiannis & Papadakis 2011a; Majorana 1981).

The G4BP also has two stable quasi-equilibrium solutions corresponding to these two equilibrium points. The initial conditions are given for one of the two quasi-equilibrium solutions as

$$\begin{aligned}
 m_1 &= 0.01, \quad m_2 = 0.021, \quad m_3 = 0.969, \quad m_4 = 1 \times 10^{-12}, \\
 \mathbf{p}'_1 &= (-6.3263361598479256638 \times 10^{-15}, 0.97966882159151619201 \times 10^{-2}), \\
 \mathbf{p}'_2 &= (-0.17988477894964014249 \times 10^{-1}, 0.97372405753200600321 \times 10^{-2}), \\
 \mathbf{p}'_3 &= (0.17988477894362678026 \times 10^{-1}, -0.19533928791999029945 \times 10^{-1}), \\
 \mathbf{p}'_4 &= (6.3263361598479256638 \times 10^{-13}, 7.6380799288046759201 \times 10^{-13}), \\
 \mathbf{q}'_1 &= (0.97966882159151279741, 6.3263871598479246128 \times 10^{-13}), \\
 \mathbf{q}'_2 &= (0.46367812263428759741, 0.85659418547566503871), \\
 \mathbf{q}'_3 &= (-0.20158853242517002593 \times 10^{-1}, -0.18563960675296861284 \times 10^{-1}), \\
 \mathbf{q}'_4 &= (0.76380799288046499741, -0.63263361598479246128). \tag{54}
 \end{aligned}$$

We introduce a rotating frame $O - x'_{[1]}x'_{[2]}$. In this frame, the origin stays at the center of mass, and the $x'_{[1]}$ axis passes through the origin and primary m_1 . Theoretically, the position of particle m_4 , \mathbf{x}'_4 , is almost at rest in the frame as well as those of the primaries, m_1 , m_2 , and m_3 . Since the G and d-CRGNBP methods are all implicit schemes, Newton's method is necessary for solving them. Here, we give starting values by Heun's method for Newton's method and for a convergence tolerance (as measured by the norm of the difference between successive iterates) of less than 10^{-16} . In the case of $\Delta t = 0.1$, Newton's iteration in the d-CRGNBP method converges, while that in the G method does not converge. For the RK4, SI4, and d-CRGNBP methods, Figure 3 shows $|\Delta \mathbf{x}'_4(t)|$, which is the shift in $\mathbf{x}'_4(t)$ at time t from the initial position $\mathbf{x}'_4(0)$. Applying the RK4, SI4, and d-CRGNBP methods, we computed $|\Delta \mathbf{x}'_4(t)|$ in the rotating frame $O - x'_{[1]}x'_{[2]}$. We used the common time step $\Delta t = 0.1$ and integrated until $t_f = 1,000,000$. The result shown in Figure 3 indicates that the shift $|\Delta \mathbf{x}'_4(t)|$ grows with time for all of the RK4 and d-CRGNBP methods and that the lower limit of error of SI4 increases. Also, the shift $|\Delta \mathbf{x}'_4(t)|$ for the d-CRGNBP is least for the time interval $0 \leq t \leq t_f$. Consequently, the result appears that the d-CRGNBP method reproduces the equilibrium solution more precisely than the others.

4.2.3. *Stable Equilibrium Points in the G5BP*

Further, we clarify that the d-CRGNBP accurately computes two stable stationary configurations in the $1 + 4$ -body problem. Such stable configurations are given as those for $n = 4$ in the $1 + n$ -body problem with one large mass and n small masses. For an arbitrary integer $n = 2, 3, \dots$, the relations satisfied by the stationary stable configurations in the $1 + n$ -body problem are described, and, for some integers, stationary configurations are numerically obtained from these relations (Casasayas, Llibre & Nunes 1994; Cors, Llibre & Ollé 2004; Salo & Yonder 1988). The initial condition corresponds

to one of those cited (see the last column of Table III in Cors, Llibre & Ollé (2004)), as follows:

$$\begin{aligned}
m_1 &= 1.0, \quad m_2 = m_3 = m_4 = m_5 = 1.0 \times 10^{-8}, \\
\mathbf{p}'_1 &= (-0.0000000250666972, -0.0000000143699690), \\
\mathbf{p}'_2 &= (0.0000000086294301, -0.0000000050530123), \\
\mathbf{p}'_3 &= (0.0000000098113650, 0.0000000019331608), \\
\mathbf{p}'_4 &= (0.0000000066259023, 0.0000000074898207), \\
\mathbf{p}'_5 &= (-0.0000000000000003, 0.0000000099999999), \\
\mathbf{q}'_1 &= (-0.0000000143699690, 0.0000000250666972), \\
\mathbf{q}'_2 &= (-0.5053012275872134, -0.8629430112463499), \\
\mathbf{q}'_3 &= (0.1933160775788700, -0.9811365039615660), \\
\mathbf{q}'_4 &= (0.7489820652276899, -0.6625902287414661), \\
\mathbf{q}'_5 &= (0.9999999856300310, 0.0000000250614611). \tag{55}
\end{aligned}$$

Because m_i ($i = 2, \dots, 5$) is sufficiently small, the stationary configuration corresponding to the initial condition (55) is stable. In the same rotating frame $O - x'_{[1]}x'_{[2]}$ in Section 4.2.2, each position of mass m_i ($i = 1, 2, \dots, 5$), $\mathbf{x}_i(t)$ is theoretically fixed at an arbitrary time t . We define $e_{\max}(t)$ as the maximum of five differences: $|\mathbf{x}_1(t) - \mathbf{x}_1(0)|$, $|\mathbf{x}_2(t) - \mathbf{x}_2(0)|$, \dots , $|\mathbf{x}_5(t) - \mathbf{x}_5(0)|$. Theoretically, $e_{\max}(t)$ is zero; $e_{\max}(t)$ is actually nonzero because of perturbation by the small masses and the influence of the numerical error. We computed $e_{\max}(t)$ using the RK4, SI4, G, and d-CRGNBP methods (see Figure 4). We used the common time step $\Delta t = 0.01$ and integrated until $t_f = 10,000$. The result indicates that the error $e_{\max}(t)$ grows linearly with time for the RK4 method, whereas the SI4, G, and d-CRGNBP methods keep $e_{\max}(t)$ within 4×10^{-4} . Thus, only the RK4 method cannot retain the central configuration in the 1 + 4-body problem. The reason is that the errors of

both the Hamiltonian H and angular momentum \mathbf{j} increase with time t for the RK4.

4.2.4. Periodic Orbits around Equilibrium Points in the G4BP

Finally, we show that the d-CRGNBP accurately computes some quasi-periodic orbits that reduce to periodic ones in the CR4BP (Baltagiannis & Papadakis 2011b) as the mass m_4 goes to zero. Baltagiannis and Papadakis listed the initial states for the non-symmetric periodic orbits of nine families in the CR4BP, all of which are linearly stable (Baltagiannis & Papadakis 2011b). If the mass m_4 is sufficiently low, the G4BP has similar families. We give the initial state for one of the families in the G4BP as follows:

$$\begin{aligned}
 m_1 &= 0.97, \quad m_2 = 0.02, \quad m_3 = 0.01, \quad m_4 = 2 \times 10^{-17}, \\
 \mathbf{p}'_1 &= (0, 0.0256637877173266), \quad \mathbf{p}'_2 = (-0.0065465367070798, -0.0183690733882485), \\
 \mathbf{p}'_3 &= (0.0065465367070798, -0.0072947143290781), \\
 \mathbf{p}'_4 &= (3.292738 \times 10^{-19}, -9.55592100000001 \times 10^{-18}), \\
 \mathbf{q}'_1 &= (0.026457513110646, 0), \quad \mathbf{q}'_2 = (-0.9184536694124222, 0.3273268353539883), \\
 \mathbf{q}'_3 &= (-0.7294714329078083, -0.6546536707079774), \quad \mathbf{q}'_4 = (-1.55780271, 0). \quad (56)
 \end{aligned}$$

As $m_4 \rightarrow 0$, this initial state corresponds to that for family f_7 in the CR4BP, where the primaries m_1 , m_2 , and m_3 revolve in a circle at the angular velocity $\omega = 1$ (see Table 2 in Baltagiannis & Papadakis (2011b)). Under the extremely small effect of the gravity of mass m_4 , the three primaries move in approximately the same circle. Therefore, we introduce a frame $O - x'_{[1]}x'_{[2]}$ rotating around the center of mass with the angular velocity $\omega = 1$, where the $x'_{[1]}$ axis always passes through the origin and the primary m_1 . In the frame $O - x_{[1]}x_{[2]}$, the three primaries move very little, in principle. In addition, Baltagiannis and Papadakis numerically clarified that mass m_4 travels in a non-symmetric closed orbit for the initial state (56) as $m_4 \rightarrow 0$.

Applying the RK4, SI4, G, and d-CRGNBP methods, we computed the orbit of particle m_4 in this frame (see Figure 5). Using a fixed time step $\Delta t = 0.1$ for the RK4, SI4, G, and d-CRGNBP methods, we integrated over the time interval $0 \leq t \leq 10,000$. For the d-CRGNBP, particle m_4 moves along a perturbed orbit around the closed non-symmetric orbit, whereas it escapes for the RK4, SI4, and G methods. In particular, Greenspan’s energy-conserving method (G) does not give the closed orbit, even though it is the only one to precisely conserve the Hamiltonian H and angular momentum \mathbf{j} .

The results given in Sections 4.2.2, 4.2.3, and 4.2.4 show that (i) only the d-CRGNBP can compute an equilibrium solution of an elliptic G4BP and periodic orbits around equilibrium points with high accuracy, and that (ii) the conservation of both H and \mathbf{j} is not necessarily sufficient for obtaining the orbit of particle m_4 .

5. Conclusion

We applied a chain regularization method and an extension of the d’Alembert-type scheme (Betsch 2005) to the general N -body problem. Then, we presented a discrete-time chain regularization of the general N -body problem (d-CRGNBP), which includes the discrete-time general three-body problem (d-G3BP) proposed by the author (Minesaki 2013a). The d-CRGNBP is second-order accurate and theoretically keeps all the conserved quantities but the angular momentum. For $N = 3$ and 4, Figure 1 in this article and Figure 2 in (Minesaki 2013a) show that the d-CRGNBP preserves the Hamiltonian.

For $N = 3$ and 4, the numerical results demonstrate that the d-CRGNBP method is superior to the symplectic and energy-momentum methods in the following sense: only the d-CRGNBP can reproduce all the equilibrium points and periodic orbits precisely, whereas the fourth-order Runge–Kutta, symplectic, and second-order energy momentum methods

cannot reproduce all of them. In particular, for $N = 3$, the author has already proved that the d-CRGNBP method has the same equilibrium points as the original general three-body problem (Minesaki 2013b). Further, for $N = 5$, the d-CRGNBP method as well as the symplectic and energy-momentum methods can precisely give all the stable equilibrium points. In future papers, we will analytically clarify that the d-CRGNBP has the same equilibrium points in the restricted four-body problem and the circular N -body problem.

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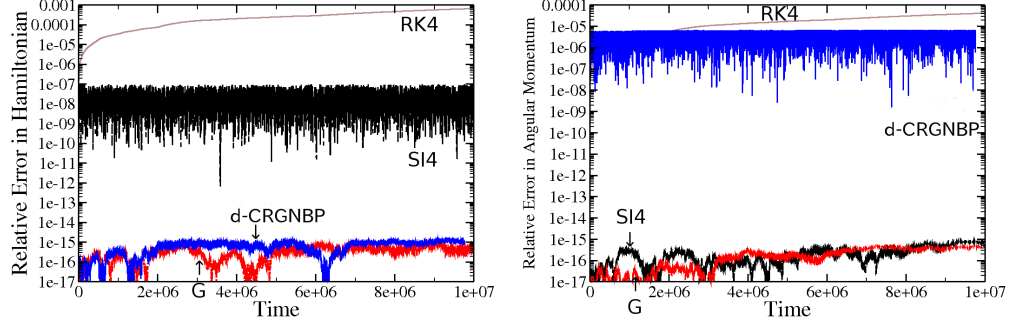


Fig. 1.— Relative errors of Hamiltonian H and angular momentum \mathbf{j} given by RK4, SI4, G, and d-CRGNBP methods. A Caledonian symmetric four-body problem is integrated (Széll, Érdi, Sándor & Steves 2004)

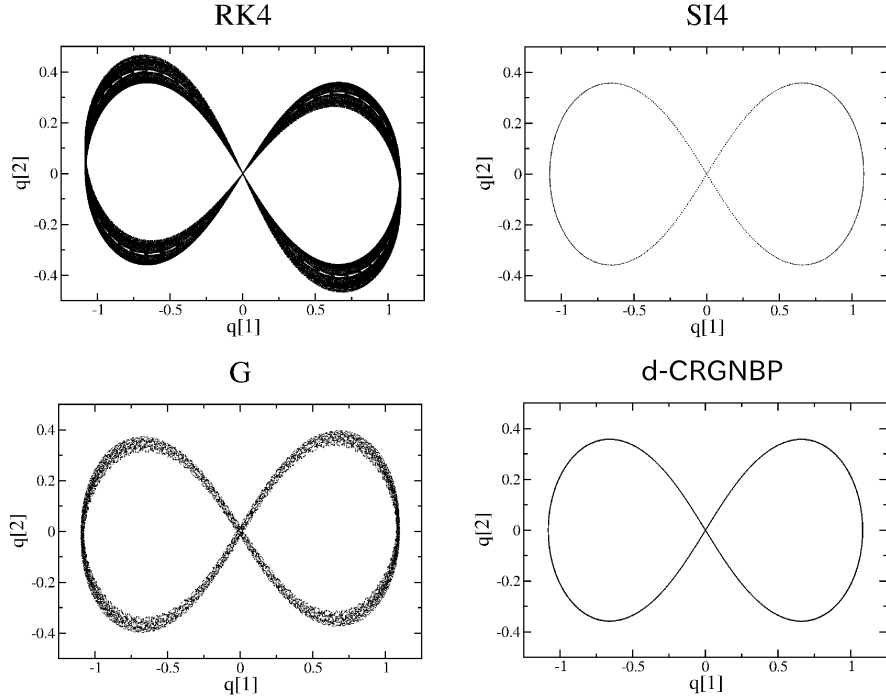


Fig. 2.— Orbits of particle m_3 for three-body figure-eight choreography (Chenciner 2000; Simó 2000) computed by RK4, SI4, G, and d-CRGNBP methods.

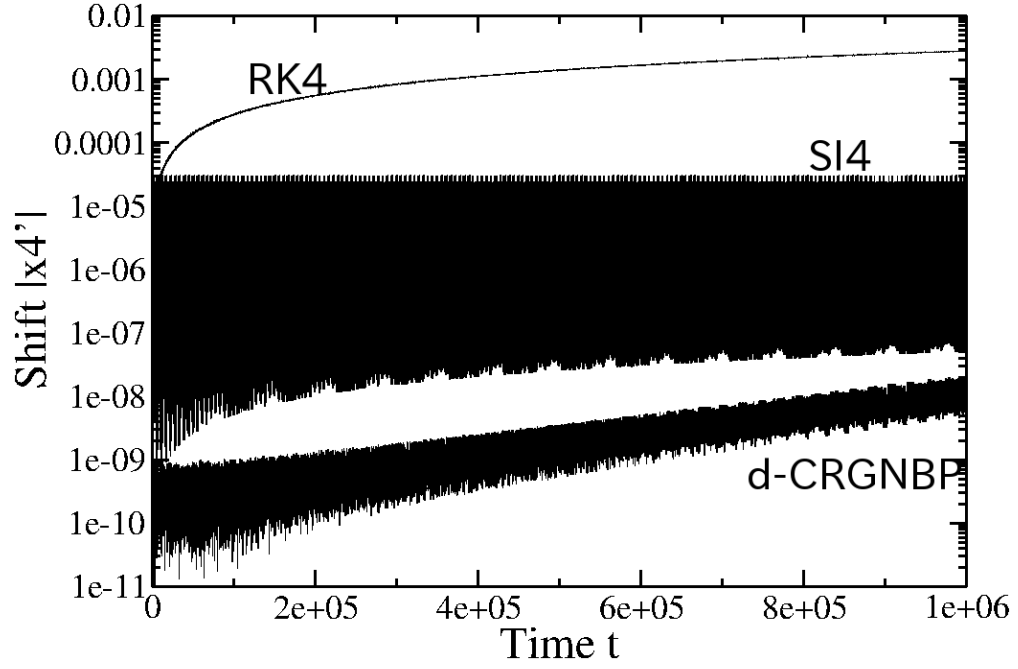


Fig. 3.— Particle m_4 shifts by $|\Delta \mathbf{x}'_4|$ from its initial position $\mathbf{x}'_4(0)$ in rotating frame $O - x'_{[1]}x'_{[2]}$ given by RK4, SI4, and d-CRGNBP methods. The position $\mathbf{x}'_4(0)$ corresponds to an equilibrium point in the G4BP. The integrated position of m_4 corresponds to the initial condition (54).

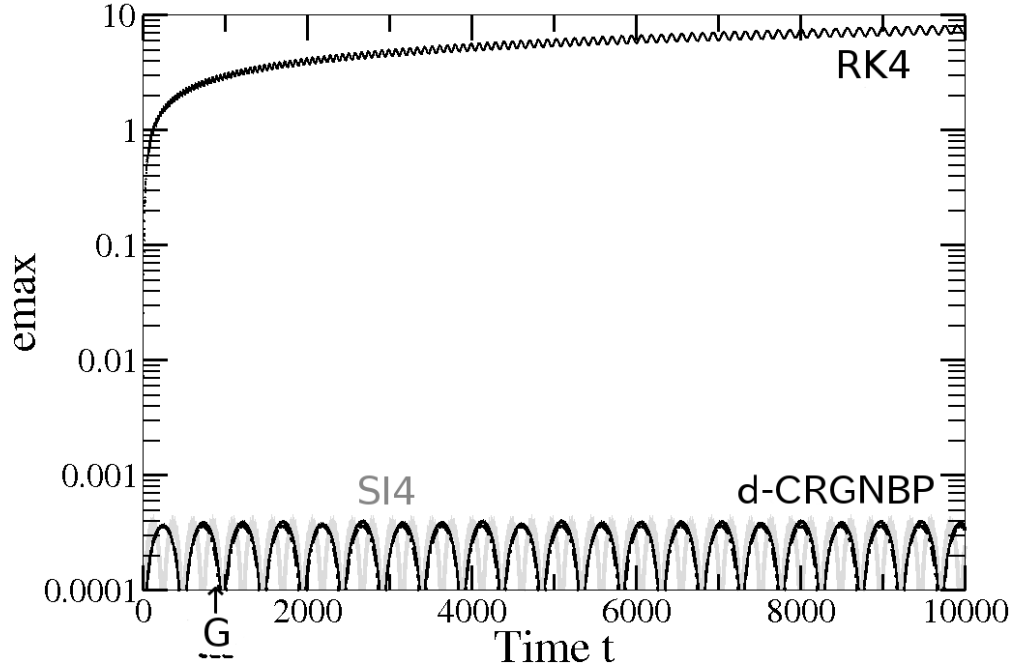


Fig. 4.— Maximum value of amplitudes of all shifts of particle m_i , $|\mathbf{x}'_i(t) - \mathbf{x}'_i(0)|$, in the rotating frame $O - x'_{[1]}x'_{[2]}$, e_{\max} given by RK4, SI4, G, and d-CRGNBP methods. Each initial position $\mathbf{x}'_i(0)$ is an equilibrium point in the $1 + 4$ -body problem. Each integrated value of $\mathbf{x}'_i(t)$ corresponds to the initial condition (55).

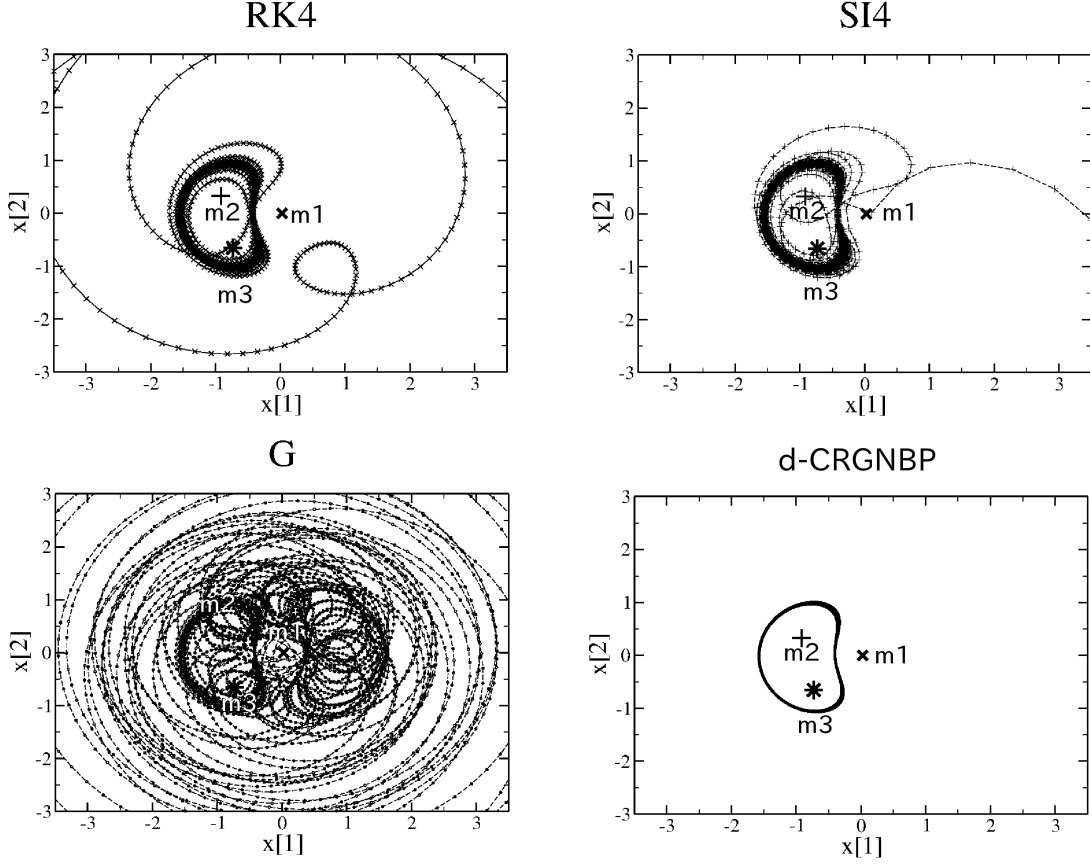


Fig. 5.— Orbits of primaries m_1 , m_2 , and m_3 in rotating frame $O - x'_1 x'_2$ given by RK4, SI4, G, and d-CRGNBP methods. The integrated orbit of particle m_4 corresponds to the initial condition (56).